Solutions to midterm 2

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Problem - 1

① Let $|\psi\rangle_{ABC}$ be a tripartite system of the form

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right).$$

$$\rho_{ABC} = |\psi\rangle\langle\psi| = \frac{1}{2}\Big[|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\Big]$$

$\mathrm{Tr}\{\rho_{ABC}^2\} = 1$, hence, ρ_{ABC} is a pure State

So, $H(\rho) = 0$.

Let's evaluate purity after tracing out system A from the tripartite state.

$$\rho_{BC} = \mathrm{tr}_A\left\{|\psi\rangle_{ABC}\langle\psi|_{ABC}\right\}$$

$$= \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$H(BC)_\rho = -\mathrm{Tr}\{\rho_{BC}\log\rho_{BC}\}$$

$$= -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$H(BC)_\rho = 1$.

Evaluation of purity: $P = \mathrm{Tr}\{\rho_{BC}^2\}$

$$\rho_{BC}^2 = \frac{1}{4}\left[|00\rangle\langle 00| + |11\rangle\langle 11|\right]$$

$$\mathrm{Tr}\{\rho_{BC}^2\} = \frac{1}{2};$$ hence, $\mathrm{Tr}\{\rho_{BC}^2\} < 1 \Rightarrow \rho_{BC}$ is a mixed state.

Similarly, for

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$$

The density matrix

$$\rho_{ABC} = \frac{1}{3}\left(|100\rangle + |010\rangle + |001\rangle\right)\left(\langle 100| + \langle 010| + \langle 001|\right)$$

$$= \frac{1}{3}\Big[|100\rangle\langle 100| + |100\rangle\langle 010| + |100\rangle\langle 001| + |010\rangle\langle 100| + |010\rangle\langle 010| + |010\rangle\langle 001| + |001\rangle\langle 100| + |001\rangle\langle 010| + |001\rangle\langle 001|\Big]$$

$$\rho_{BC} = \mathrm{Tr}_A\{\rho_{ABC}\}$$

$$= \frac{1}{3}\left[|00\rangle\langle 00| + |10\rangle\langle 10| + |10\rangle\langle 01| + |01\rangle\langle 10| + |01\rangle\langle 01|\right]$$

$$= \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\rho_{BC}^2 = \frac{1}{9}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{Tr}\{\rho_{BC}^2\} = 5/9 \Rightarrow \rho_{BC} \text{ is a mixed state.}$$

To evaluate entropy of BC system, we need to obtain its spectral form. Hence, we need to evaluate the eigen values of ρ_{BC}. The eigen values are computed using the characteristic eq.^{n}

$$|\rho_{BC} - \lambda \mathbb{I}| = 0$$

$$\Rightarrow \begin{vmatrix} 1/3-\lambda & 0 & 0 & 0 \\ 0 & 1/3-\lambda & 1/3 & 0 \\ 0 & 1/3 & 1/3-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda\left(\lambda - \tfrac{1}{3}\right)\left[\left(\tfrac{1}{3} - \lambda\right)^2 - \tfrac{1}{9}\right] = 0$$

$$\Rightarrow \lambda = 0, 0, 1/3, 2/3$$

$$\rho_{BC}^2 = \frac{1}{9}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{Tr}\{\rho_{BC}^2\} = 5/9 \Rightarrow \rho_{BC} \text{ is a mixed state.}$$

$$\text{So, } H(BC)_\rho = -\tfrac{1}{3}\log\tfrac{1}{3} - \tfrac{2}{3}\log\tfrac{2}{3}$$

$$= \tfrac{1}{3}\log 3 + \tfrac{2}{3}\log 3/2$$

$$= 0.918$$

(2) Prove that monotonicity of trace distance & fidelity under quantum channel action N.

Proof:

We have to show —

$$\| N(\rho) - N(\sigma) \|_1 \leq \|\rho - \sigma\|_1$$

A quantum channel can be expressed as a isometric evolution acting on a larger Hilbert space, such as

$$(\rho^S \otimes |0\rangle\langle 0|^E)$$

We know that trace distance is invariant under isometric evolutions

$$\|\rho - \sigma\|_1 = \| \rho^S \otimes |0\rangle\langle 0|^E - \sigma \otimes |0\rangle\langle 0|^E \|_1$$

$$\left[\text{As } \|\rho \otimes \omega - \sigma \otimes \omega\|_1 = \|\rho - \sigma\|_1 \right]$$

$$= \| U(\rho \otimes |0\rangle\langle 0|)U^\dagger - U(\sigma \otimes |0\rangle\langle 0|)U^\dagger \|$$

[Trace distance is invariant under isometric evolution]

Now, tracing out the environmental subsystem, we have

$$\geq \| Tr_E \{ U(\rho \otimes |0\rangle\langle 0|^E)U^\dagger - U(\sigma \otimes |0\rangle\langle 0|^E)U^\dagger \|_1$$

$$= \|N(\rho) - N(\sigma)\|_1 \quad \boxtimes$$

Similarly, we have to show that,

$$\mathbb{F}(\rho,\sigma) \leq \mathbb{F}(\mathcal{N}(\rho), \mathcal{N}(\sigma))$$

Consider, $\mathbb{F}(\rho,\sigma) = \mathbb{F}(U\rho U^{\dagger}, U\sigma U^{\dagger})$ [$U^{A\to BE}$ be an isometric extension]

$$= \mathbb{F}\left(U(\rho \otimes |0\rangle\langle 0|)U^{\dagger}, U(\sigma \otimes |0\rangle\langle 0|)U^{\dagger}\right)$$

[U is operating on BE]

So, $\mathbb{F}(\rho,\sigma) = \mathbb{F}(\rho \otimes |0\rangle\langle 0|, \sigma \otimes |0\rangle\langle 0|)$

$$= \mathbb{F}(\rho,\sigma)\, \mathbb{F}(|0\rangle\langle 0|, |0\rangle\langle 0|)$$

So, $\mathbb{F}(\rho,\sigma) \leq \mathbb{F}\left(\operatorname{tr}_E\left(U(\rho \otimes |0\rangle\langle 0|)U^{\dagger}\right), \operatorname{tr}_E\left(U(\sigma \otimes |0\rangle\langle 0|)U^{\dagger}\right)\right)$

$$= \mathbb{F}(\mathcal{N}(\rho), \mathcal{N}(\sigma))$$ [P]

Show that entanglement fidelity is convex under quantum channel $\mathcal{N}$

Proof:- :-

We have to show that

$$F_e(\lambda \rho_1 + (1-\lambda)\rho_2, \mathcal{N})$$
$$\leq \lambda F_e(\rho_1, \mathcal{N}) + (1-\lambda) F_e(\rho_2, \mathcal{N})$$

From the theorem taught in the class, we know that a quantum channel $\mathcal{N}$ with Kraus operators A_m, the entanglement fidelity is given by

$$F_e(\rho, \mathcal{N}) = \sum_m \left| Tr\{\rho^A A_m\} \right|^2 \quad -(1)$$

Hence,

$$F_e(\lambda \rho_1 + (1-\lambda)\rho_2, \mathcal{N})$$

$$= \sum_m \left| Tr\left\{ (\lambda \rho_1 + (1-\lambda)\rho_2)^A A_m \right\} \right|^2$$

$$= \sum_m \left| Tr\left\{ (\lambda \rho_1)^A A_m + ((1-\lambda)\rho_2)^A A_m \right\} \right|^2$$

$$= \sum_m \left| Tr\left\{ (\lambda \rho_1)^A A_m \right\} + Tr\left\{ ((1-\lambda)\rho_2)^A A_m \right\} \right|^2$$

$$= \sum_m \left| \lambda Tr\{\rho_1 A_m\} + (1-\lambda) Tr\{\rho_2 A_m\} \right|^2$$

$$\leq \sum_m \lambda \left| Tr\{\rho_1 A_m\} \right|^2 + (1-\lambda) \left| Tr\{\rho_2 A_m\} \right|^2$$

[This inequality holds for convexity of Σ]

Hence,

$$\mathbb{F}_e(\lambda \rho_1 + (1-\lambda)\rho_2 , N) \leq \lambda \, \mathbb{F}_e(\rho_1, N) + (1-\lambda)\, \mathbb{F}_e(\rho_2, N)$$

$\blacksquare$

Problem - 2

① For an ensamble $\mathcal{E}_0 = \{ P_X(x), |\psi_x\rangle \}$, Show that

$$H(\rho) \leq H(X),$$

$p_i^0 \equiv P_X(x)$, $|\psi_x\rangle\langle\psi_x| \equiv \rho_i^0$

$$\text{mixture} = \sum_i p_i^0 \rho_i^0 \equiv \sum_{x \in X} P_X(x) |\psi_x\rangle\langle\psi_x|$$

Proof:

Let us begin with a pure state $\rho_x = |\psi_x\rangle\langle\psi_x|$.

Let ρ_x be states of system A. Introduce anchron System R with orthonormal basis $|x\rangle$ corresponding to index x over $P_X(x)$.

Define $|AR\rangle \equiv \sum_{x \in X} \sqrt{P_X(x)} \, |\psi_x\rangle |x\rangle$.

As $|AR\rangle$ is a pure state, we evaluate Von Neumann entropy as

$$S(A) = S(R) = S\left(\sum_{x \in X} P_X(x) |\psi_x\rangle\langle\psi_x| \right) = S(\rho)$$

Let us perform projective measurements on system R on $|x\rangle$ basis. Post measurements, the state of the system R is

$$\rho^{R'} = \sum_{x \in X} P_X(x) |x\rangle\langle x|.$$

Refer Theorem: 11.9 (Nielsen Chuang Book), Projective measurement increases entropy.

Hence, $S(\rho) = S(R) \le S(R') = H(\mathbb{P}_X(x))$

Note that
$$S(\rho) \le H(P_X) + \sum_x P_X(x) S(\rho_x)$$
where ρ_x's are pure states.

Furthermore, the above result holds when $|\psi_x\rangle$ are orthogonal to each other.

Consider a mixed state given by
$$\rho_x = \sum_{y \in Y} P_Y(y)_x |y\rangle_x \langle y|_x,$$ over orthonormal decomposition of ρ_x; hence,
$$\rho = \sum_{x \in X, y \in Y} P_X(x) P_Y(y)^x |y\rangle_x \langle y|_x \quad \text{--- (1)}$$

We know that $\sum_y P_Y(y) = 1$ for each $x \in X$.

Hence, we have

$$S(\rho) \leq -\sum_{x \in X, y \in Y} P_X(x) P_Y(y)_x \log \{ P_X(x) P_Y(y)_x \}$$

$$= -\sum_{x \in X} P_X(x) \log P_X(x) - \sum_{x \in X} P_X(x) \sum_{y \in Y} P_Y(y)_x \log P_Y(y)_x$$

$$= H(P_X(x)) + \sum_{x \in X} P_X(x) S(\rho_x).$$

$\blacksquare$

Appendix :- A (Projective measurement increases entropy)

Proof:-

Let Π_x be a complete set of orthogonal projectors, and ρ is the density operator. We need to show that $\rho' = \sum_x \Pi_x \rho \Pi_x$ after measurements is atleast greater than the original entropy.

$$S(\rho') \geq S(\rho)$$

Proof:- We know that relative entropy $S(\rho' \| \rho) \geq 0$
&

$$S(\rho' \| \rho) = -S(\rho) - \mathrm{tr}(\rho \log \rho')$$

$$\Rightarrow -S(\rho) - \mathrm{tr}(\rho \log \rho') \geq 0$$

$$\Rightarrow \quad S(\rho) \le -\mathrm{tr}\,(\rho \log \rho')$$

If we show that $-\mathrm{tr}\,(\rho\log\rho') = S(\rho')$, we can prove the above theorem.

As

$$\sum_x \Pi_x = I \quad \& \quad \Pi_x^2 = \Pi_x ,$$

$$-\mathrm{tr}\left\{\rho\log\rho'\right\}$$

$$= -\mathrm{tr}\left\{\sum_x \Pi_x \rho \log\rho'\right\}$$

$$= -\mathrm{tr}\left\{\sum_x \Pi_x^2 \rho\log\rho'\right\} \quad (\Pi_x = \Pi_x^2)$$

$$= -\mathrm{tr}\left\{\sum_x \Pi_x \rho \log\rho' \,\Pi_x\right\} \quad (\text{cyclic property of trace})$$

As $\rho' = \sum_x \Pi_x \rho \Pi_x$

$$\Rightarrow \rho'\Pi_x = \sum_x \Pi_x \rho \Pi_x^2 = \Pi_x \rho'$$

$$\Rightarrow \quad [\Pi_x, \rho'] = 0 \Rightarrow [\Pi_x, \log\rho'] = 0$$

Π_x commutes with ρ' $\Rightarrow$ Π_x also commutes with $\log\rho'$

$$\Rightarrow \quad -\mathrm{tr}\left\{\rho\log\rho'\right\} = -\mathrm{tr}\left\{\sum_x \Pi_x \rho \Pi_x \log\rho'\right\}$$

$$= -\mathrm{tr}\left\{\rho'\log\rho'\right\} = S(\rho')$$

$\blacksquare$

(2) Any quantum operation in the same Hilbert space can be visualised as a unitary evolution denoted by U.

Assume that we have a noisy quantum state $\rho_{mix} \in \mathbb{C}^2$.

We want to purify in the same Hilbert space $\mathbb{C}^2$ by performing quantum operation $(U \rho_{mix} U^\dagger) = \rho_{mf}$

Its purity $P_1 = \mathrm{Tr}\{\rho_{mix}^2\}$ in $\mathbb{C}^2$

After evolution purity $\equiv P_2 = \mathrm{Tr}\{(U \rho_{mix} U^\dagger)^2\}$

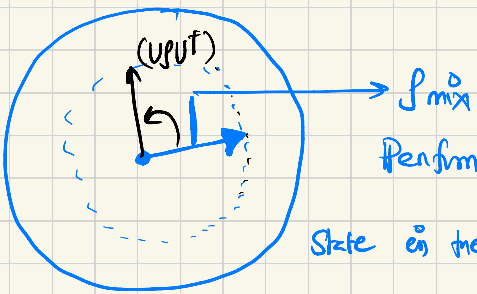
$$= \mathrm{Tr}\{U \rho_{mix} U^\dagger U \rho_{mix} U^\dagger\}$$

$$= \mathrm{Tr}\{U \rho_{mix}^2 U^\dagger\}$$

$$= \mathrm{Tr}\{\rho_{mix}^2\} = P_1 \quad \blacksquare$$

Purity of the state remains the same for any quantum operation in the same Hilbert space.

Geometrical interpretation:-



Consider a mixed state in $\mathbb{C}^2$,

Performing a quantum operation on a mixed state in the same Hilbert space is equivalent to performing a unitary evolution.

Applying unitary operation on any Bloch vector in $\mathbb{C}^2$ only rotates the vector in the Bloch sphere rather than increasing its length.

Hence, performing any unitary operation on the same Hilbert space will not increase the purity of a given noisy quantum state.

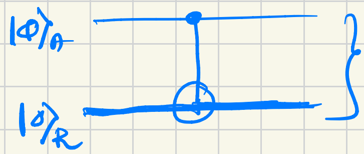
Purification can be done by bringing in a known reference state so as to increase the dimension of joint Hilbert space.

We have a reference pure state as $|\psi\rangle_R = |0\rangle_R$ in $\mathbb{C}^2$. Consider another state $|\phi\rangle_A = \dfrac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}$

So, the joint state becomes

$$|\phi\rangle_{AR} = \frac{|00\rangle_{AR} + |10\rangle_{AR}}{\sqrt{2}}.$$

Applying a CNOT gate :-



$$|\Phi^+\rangle \Downarrow \frac{|00\rangle_{AR} + |11\rangle_{AR}}{\sqrt{2}} \Downarrow |\Phi^+\rangle$$

We get $\rho_A = \mathrm{Tr}_R\{|\Phi^+\rangle\langle\Phi^+|\}$.

$$\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)$$

$$\mathrm{Tr}_R\left[\frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2}\right] = \frac{I}{2}$$

Reversibility of tracing operation is impossible. Hence, to construct a Bell state from a mixed state is impossible. However, adding ancilla from a different system denoted by R, we can transform a mixed state to a Bell state.

③ $$\mathbb{F}(\rho^{(1)}, |\psi\rangle) \leq \mathbb{F}(\rho^{(2)}, |\psi\rangle) + \|\rho^{(1)} - \rho^{(2)}\|_1$$

Proof:

Consider a projector $\Pi = |\phi\rangle\langle\phi|$

We know that

$$\|\rho^{(1)} - \rho^{(2)}\|_1 = \max_{0 \leq \Lambda \leq \mathbb{I}} \mathrm{Tr}\{\Lambda(\rho^{(1)} - \rho^{(2)})\}.$$

Now, $\mathrm{Tr}\{\Pi(\rho^{(1)} - \rho^{(2)})\} \leq \|\rho^{(1)} - \rho^{(2)}\|_1$

[As Π may not be a maximizing POVM]

$$\Rightarrow \mathrm{Tr}\{\Pi(\rho^{(1)})\} \leq \mathrm{Tr}\{\Pi(\rho^{(2)})\} + \|\rho^{(1)} - \rho^{(2)}\|_1$$

$$\Rightarrow \mathrm{Tr}\{|\phi\rangle\langle\phi|\rho^{(1)}\} \leq \mathrm{Tr}\{|\phi\rangle\langle\phi|\rho^{(2)}\} + \|\rho^{(1)} - \rho^{(2)}\|_1$$

$$\Rightarrow \mathrm{Tr}\{\langle\phi|\rho^{(1)}|\phi\rangle\} \leq \mathrm{Tr}\{\langle\phi|\rho^{(2)}|\phi\rangle\} + \|\rho^{(1)} - \rho^{(2)}\|_1$$

$$\Rightarrow \mathbb{F}(\rho^{(1)}, |\phi\rangle) \leq \mathbb{F}(\rho^{(2)}, |\phi\rangle) + \|\rho^{(1)} - \rho^{(2)}\|_1$$

$\blacksquare$

③ ① Prove that $I(A,B) \le 2\min\left(\log(d_A), \log(d_B)\right)$

Where d_A, d_B are dimension of quantum system A & B respectively.

Solution:

We know that $I(A,B)_\rho = H(A)_\rho + H(B)_\rho - H(AB)_\rho$

$$= H(A)_\rho - H(A|B)_\rho$$

$$= H(B)_\rho - H(B|A)_\rho$$

we know that $H(A|B) \le H(A)$

$$\le \log d_A \Rightarrow |H(A|B)| \le \log d_A$$

$$\Rightarrow -H(A|B) \le \log d_A.$$

Similarly, $H(B|A) \le H(B)$

$$\le \log_2 d_B \Rightarrow |H(B|A)| \le \log d_B$$

$$\Rightarrow -H(B|A) \le \log d_B$$

So, $I(A,B) = H(A)_\rho - H(A|B)_\rho$

$$\le \log d_A + \log d_A$$

$$\le 2\log d_A \qquad — ①$$

Further, $I(A,B) = H(B) - H(B|A)$

$$\le 2\log d_B. \qquad — ②$$

So, Combining ① & ②

$$I(A;B) \le 2\min\left(\log d_A, \log d_B\right).$$

② Consider the action of isometry $U^{A\to BE}$ on tripartite state $|\psi\rangle^{SRA}$ to produce $|\phi\rangle^{SRBE}$. Show that

$$I(R;A)_{|\psi\rangle} + I(R;S)_{|\psi\rangle} = I(R;B)_{|\phi\rangle} + I(R;SE)_{|\phi\rangle}$$

Solution:-

Let $|\psi\rangle^{SRA}$ be a pure state. We know that

$$\left.\begin{aligned} H(SRA) &= 0 \\ H(S) &= H(RA) \\ H(R) &= H(SA) \\ H(A) &= H(SR) \end{aligned}\right\} \text{By applying Bipartite cuts.}$$

$$\begin{aligned} I(R;A) &= H(R) + H(A) - H(RA) \\ &= H(R) + H(A) - H(S) \quad \text{— ①} \end{aligned}$$

$$\begin{aligned} I(R;S) &= H(R) + H(S) - H(RS) \\ &= H(R) + H(S) - H(A). \quad \text{— ②} \end{aligned}$$

Adding ① & ②, we get

$$\begin{aligned} &I(R;A) + I(R;S) \\ &= H(R) + \cancel{H(A)} - \cancel{H(S)} + H(R) + \cancel{H(S)} - \cancel{H(A)} \\ &= 2H(R) \end{aligned}$$

RHS :- $I(R;B) + I(R;SE)$

$$= H(R) + H(B) - H(RB)$$
$$+ H(R) + H(SE) - H(RSE) \quad \text{— (3)}$$

Let $|\phi\rangle^{SRBE}$ be a pure state post isometry over A.

Applying bipartite cuts over the system SRBE, we get

$$\left.\begin{aligned} H(RSE) &= H(B) \\ H(RB) &= H(SE) \end{aligned}\right\}$$

$$H(R) + \cancel{H(B)} - \cancel{H(SE)} + H(R) + \cancel{H(SE)} - \cancel{H(B)}$$

$$= 2H(R) = LHS.$$

So, the action of isometry $U^{A\to BE}$ on tripartite state $|\psi\rangle^{SRA}$ to produce $|\phi\rangle^{SRBE}$ yields

$$I(R;A)_{|\psi\rangle} + I(R;S)_{|\psi\rangle} = I(R;B)_{|\phi\rangle} + I(R;SE)_{|\phi\rangle}$$

$\blacksquare$