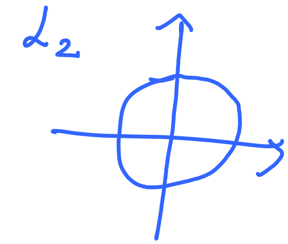
# Basics of L1 Regularization




## Issues we may face with $L_2$ regularization:

Consider the case where we have 2 variables in a weight vector, say $w_1$ and $w_2$ that are highly correlated.

If $w_1 \uparrow\uparrow$, $w_2 \downarrow\downarrow$, in a way, canceling the effect of $w_1$ $\implies$ models that can have high variance, implying different predictions under the RSS (residual sq. sum)

We considered the regression problem earlier

$X := [x_{ij}]_{n \times k}$ $\quad \underline{x}_i \in \mathbb{R}^k$

$\underline{y}_{n \times 1}$ data response

$$\underline{w} = \left(X^T X + \lambda I\right)^{-1} X^T \underline{y} \qquad (\text{min-norm soln})$$

Cost functional: $\| X \underline{w} - \underline{y} \|_2^2 + \lambda \| \underline{w} \|_2^2$

(a) $\underline{w}$ must not be too large

(b) Balance the large variables in $\underline{w}$ meeting the target cost.

$L_2$ norm

1) does not account for the parsimony of the model
i.e., sparsity constraints are not taken into account.

2) $L_2$ models may have non-zero values associated with inconsequential variables.

Costs involving $L_1$ penalty impose sparsity constraints $\|\underline{w}\|_1$

1) If the data matrix $X_{(n \times k)}$ has irrelevant features, $L_1$ seems to be better than $L_2$ $\Longrightarrow$ low variance feature selection

2) $L_1$ can yield a better variable/attribute selection

a) Simplification of models for interpretability.

b) Shorter training times

c) Avoid the problem of overfitting $\Longrightarrow$ avoid the curse of dimensionality

## Unconstrained formulation

$$\min_{\underline{w}} \; \|X\underline{w} - \underline{y}\|_2^2 + \lambda \|\underline{w}\|_1 \qquad (1)$$

$L_1$ norm

Clearly (1) has issues of differentiability @ the origin

$$\|w\|_1 = |w_1| + |w_2| + \cdots + |w_k|$$

$|x|$ is not differentiable @ $x = 0$.

# Constrained Formulation

$$\min_{\underline{w}} \; \| X\underline{w} - \underline{y} \|_2^2 \quad \text{s.t.} \quad \|\underline{w}\|_1 \leq t \quad \text{--- } (2)$$

chosen value

Non-differentiable constraints are converted to a set of linear constraints

$\Rightarrow$ Feasible region is a polyhedron!

# ALGORITHMS

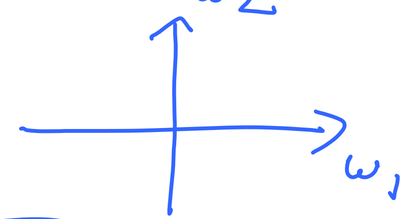
## LASSO

Least Absolute Selection and Shrinkage Operator.

Consider solving the problem (2).

Suppose we have 2 variables in $\underline{w}$, say $w_1$ and $w_2$

By considering the sign of $w_1$ and $w_2$




$$\left.\begin{aligned} w_1 + w_2 &\le t\\ w_1 - w_2 &\le t\\ -w_1 + w_2 &\le t\\ -w_1 - w_2 &\le t \end{aligned}\right\} \quad \mathrm{I}$$

## Home Work

Plot the constraints on $w_1 - w_2$ plane and show the feasible region.

Any minimizer to the RSS subject to (I) will minimize the cost (2)

Problem : If we have 'k' variables, we have $2^k$ constraints $\implies$ exponential increase in complexity.

Over $\mathbb{R}^{40}$, $2^{40}$ are possible (Infeasible for optimization)

# TIBSHIRANI'S APPROACH

(In practice 't' can be small)

Constraint Set $= \phi$

while $(\|\underline{w}\|_1 \leq t)$

- Add sign$(\underline{w})$ and fold this into the constraint set.
- Opt $\|X\underline{w} - \underline{y}\|_2^2$ subject to the constraints.

end while

Instead of testing if $\|\underline{w}\|_1 \leq t$, we can introduce $\varepsilon > 0$ / we consider $\|\underline{w}\|_1 \leq t + \varepsilon$

At every iteration, $\|\underline{w}\|_1$ shrinks

1) The soln from a previous iteration may not be suited to the present constraint $\Rightarrow$ One needs to optimize again

2) Adding the sign constraints can have variables that can have large swings from +ve / -ve. (Correlated variables)

## Introduce non-negative variables

**Idea:** Express each $w_i$ as a difference of two non-negative variables

$$w_i = w_i^+ - w_i^- \qquad \left( \begin{array}{l} w_i^+ \geq 0 \\ w_i^- \geq 0 \end{array} \right)$$

$$\left. \begin{array}{l} \text{If } w_i > 0; \quad w_i^+ = w_i; \; w_i^- = 0 \\ w_i < 0; \quad w_i^+ = 0; \; w_i^- = w_i \\ w_i^+ = w_i^- = 0 \text{ if } w_i = 0 \end{array} \right\}$$

For $k$ variables in $\underline{w}$, we introduce $2k$ non negative variables $\Rightarrow$ introducing degeneracy in the constraints

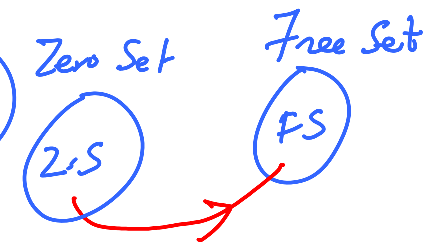
For e.g., if we consider a 2 variable case

$$w_1^+ \geq 0 \qquad w_1^- \geq 0$$

$$w_2^+ \geq 0 \qquad w_2^- \geq 0$$

$$\sum_{i=1}^{2} \left( w_i^+ + w_i^- \right) \leq t$$

__Grafting__ : (Perkins et al, JMLR)




__Idea:__ Incrementally build a subset of param. allowed to differ from 0s.

At each iteration, we use a fast grad. meta heuristic to decide which zero wt. should be adjusted away from zero to decrease the opt. criterion by max. amount

Recall :  $\|X\underline{w} - y\|^2 + \lambda \|\underline{w}\|_1$

$$\nabla_{\underline{w}} = X^T(y - X\underline{w}) + \lambda \,\mathrm{sign}(\underline{w})$$

For variables that are zero, $w_i = 0$

$\mathrm{sign}(w_i) = 1$ if $\underline{X}_{i\,{}_{1\times n}}^T (\underline{y}_{n\times 1} - X_{n\times k}\underline{w}_{k\times 1}) > \lambda$

$i$ th data point
i.e., attribute 'i' over all data

By convention, if $\underline{X}_i^T (y - X\underline{w}) = \lambda$, grad is set to 0

A procedure can be evolved as follows

1) Consider all variables in the zero set initially

2) At each iteration, test if

$$\left| X_i^T (y - X\underline{w}) \right| \le \lambda \quad \text{for each } 'i' \qquad (A)$$

If Condition (A) is false, the variable whose derivative has the largest mag. is added to the free set.

3) Any popular method (QN, BFGS algo) can be used to optimize the variables in the free set.

# Discussion on VC-dimension

**Defn:** A dichotomy of a set 'S' is a partition of S into 2 disjoint subsets

$$S = \{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_{100}\}$$

$$S_\times = \{\underline{x}_1, \underline{x}_3, \ldots, \underline{x}_{99}\}$$

$$S_o = \{\underline{x}_2, \underline{x}_4, \ldots, \underline{x}_{100}\}$$

(2-class problem)

<u>Defn</u> : A set of instances S is '<u>shattered</u>' by a hypothesis space H <u>iff</u> for every dichotomy of S $\exists$ a hypothesis that is consistent with the dichotomy

$\exists\, h \in H$ / $h_1$ classifies $S_x$ as +ve $\hookrightarrow$ '1'

$h_2$ —"— $S_o$ as −ve $\hookrightarrow$ '2'

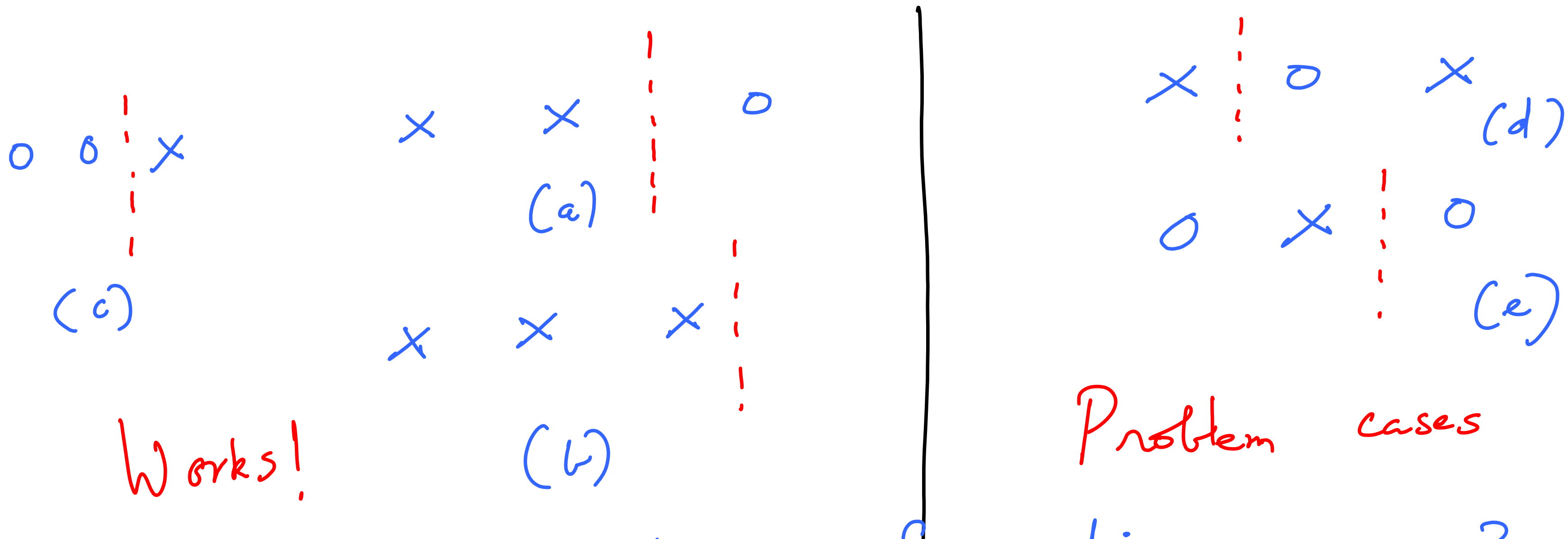
## Motivate the notion of VC-dimension

Consider points on a line (points $\in S_x$ or $S_o$)


2 points on a line

"Can shatter the 2 points on a line"

Let us consider 3 points on a line

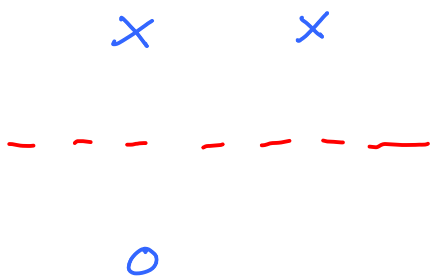


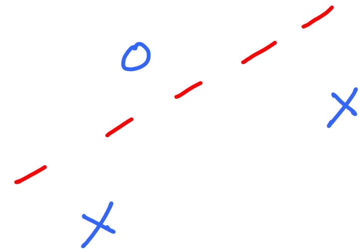

Works!

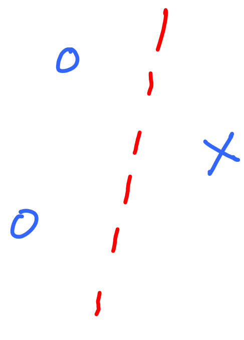
Problem cases

"8 different configurations over 3 points on a line"

Let us consider points in $\mathbb{R}^2$ i.e., on a plane (3 points case)

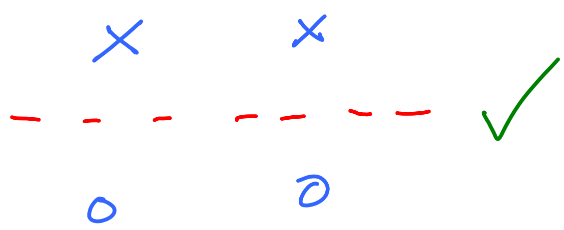


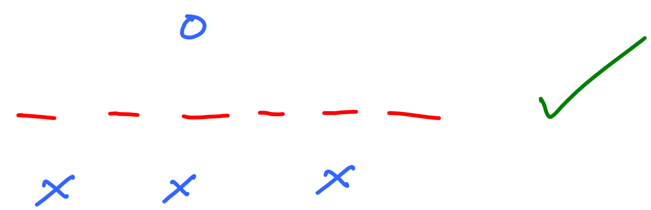


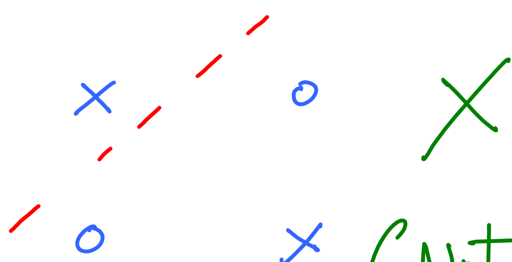


3 points can be shattered in $\mathbb{R}^2$

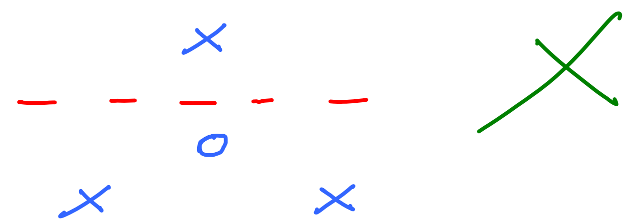
Let us increase by '1' extra point i.e., 4 points in $\mathbb{R}^2$







(Not possible)



(Not possible)

If I consider a hyperplane for shattering points over d- dimensions,

$$\underline{w}^T \underline{x} = 0$$

Expanding out

$$\sum_{i=1}^{d} w_i x_i + w_0 \cdot 1 = 0$$

to accommodate bias

projection of d features on to a vector normal to the plane

# points that can be shattered in $\mathbb{R}^d$ is

$\underline{\underline{d+1}}$ !

VC (H)

**Defn :** The VC dimension of a hypothesis space H defined over a data set X is the size of the largest subset of X shattered by H

The reader can refer to the PAC bound derivation (probably approximately correct) in any standard M.L. text book.

For linear classifiers with $\underline{x} \in \mathbb{R}^d$

$VC(\mathcal{H}) = d+1$ ; $d$ : # of features

For neural networks

$VC(\mathcal{H}) =$ # parameters in the n/w

$\propto$ # of neurons in a hidden layer

for a single hidden layer MLP.

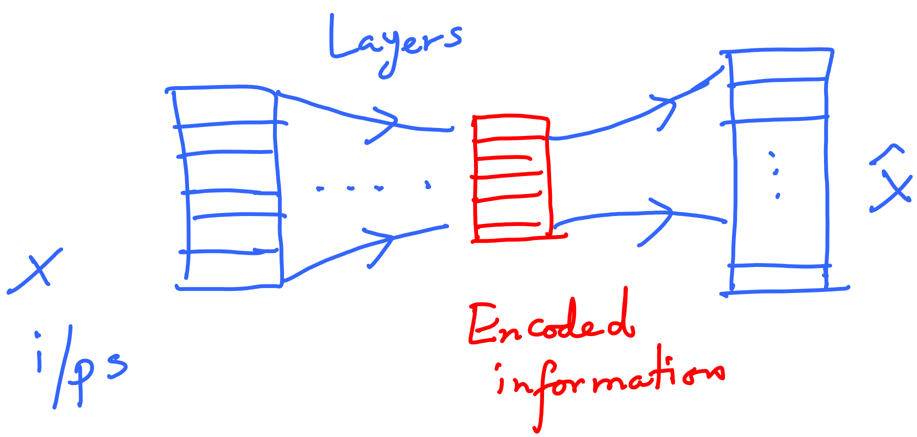
# Auto Encoders

This is a neural network for data encodings and to learn a representation of a data vector in a reduced dimension to ignore signal "noise"

**Idea:** We have a sequence of layers from the i/p to learn local features all the way to less local features and eventually the object.

Decoding layers will decode the learnt encoded information

## Structure




Typically, we use feed forward N/w such as MLP

O/P has the same # nodes as the input

Let $\phi_E$ and $\phi_D$ be non-linear mappings

$$\phi_E : \mathcal{X} \longrightarrow \mathcal{F}$$

$$\phi_D : \mathcal{F} \longrightarrow \mathcal{X}$$

$\underline{x} \in \mathbb{R}^d = \mathcal{X}$

$\underline{z} \in \mathbb{R}^k = \mathcal{F}$

$k < d$

$$\phi_E^*, \phi_D^* = \underset{\phi_E, \phi_D}{\arg\min} \, E \left\| x - (\phi_D \circ \phi_E) x \right\|^2$$

$\longleftarrow$ order of composition

For purposes of simplicity let us consider a single hidden layer

$$\underline{z} = \sigma\left(\underline{w}^T \underline{x} + b\right)$$

sigmoid / ReLU

coded representation / latent representation

Reconstruction

$$\underline{\hat{x}} = \hat{\sigma}\left(\underline{\hat{W}}^T \underline{z} + \hat{b}\right)$$

Loss function

$$\mathcal{L}(\underline{x}, \underline{\hat{x}}) = \left\| \underline{x} - \underline{\hat{x}} \right\|^2$$

$$= \left\| \underline{x} - \underbrace{\hat{\sigma}\left(\underline{\hat{W}}^T\left(\sigma\left(\underline{W}^T \underline{x} + b\right)\right) + \hat{b}\right)}_{\underline{\hat{x}}} \right\|^2$$

One can also minimize the "average loss" computed by taking the $E(\cdot)$ over $\mathcal{L}(\underline{x}, \underline{\hat{x}})$

# Denoising auto encoder

**Idea** : Take a partially corrupted input during training to recover the original undistorted i/p data vector.

**Hope** : We have an efficient representation that can robustly obtain the clean i/p from a corrupted representation!

## Assumptions (JMLR, Vincent etal)

1) Higher level representations are robust and stable

2) Extract features that are useful to represent the original data (pdf).

## Stochastic corruption

$S_c : \underline{X} \longrightarrow \underline{\tilde{X}}$

Feed $\{\underline{\tilde{X}}\}$ to the auto encoder for learning

Loss is $L(\underline{x}, \underline{\hat{\tilde{x}}})$

decoded o/p based on the encoded version of the corrupted input (detail)

original uncorrupted i/p

## Sparsity constraints with the N.N.

More hidden units than i/ps
but very few of them could be "active"

One can bring in "regularization" constraints

# Geometric Interpretation

$q_D(\tilde{x}|x)$ $\qquad$ $g_{\theta'}(f_\theta(\tilde{x}))$

Interim representation

$\Downarrow$

Interpret it as a coord. sys. for points $\underline{x}$ on the manifold

$\dim(f_\theta(\cdot)) < \dim(\underline{x})$