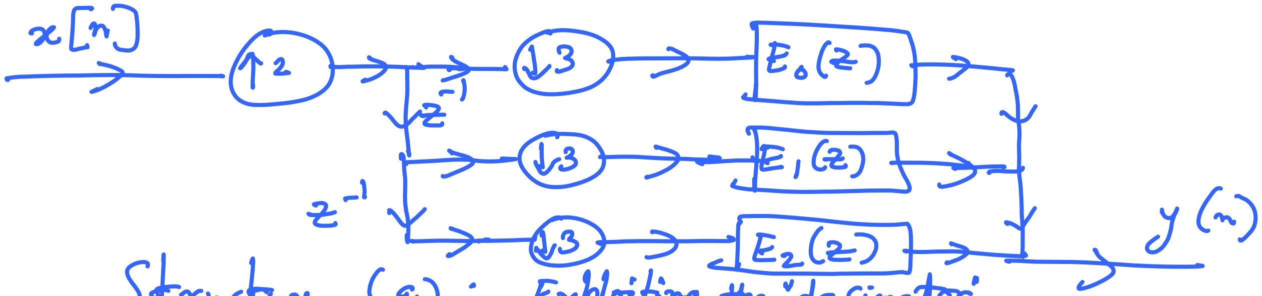
Efficient structures for fractional decimation

For doing a M/L decimation, w/o polyphase we are 'inefficient'

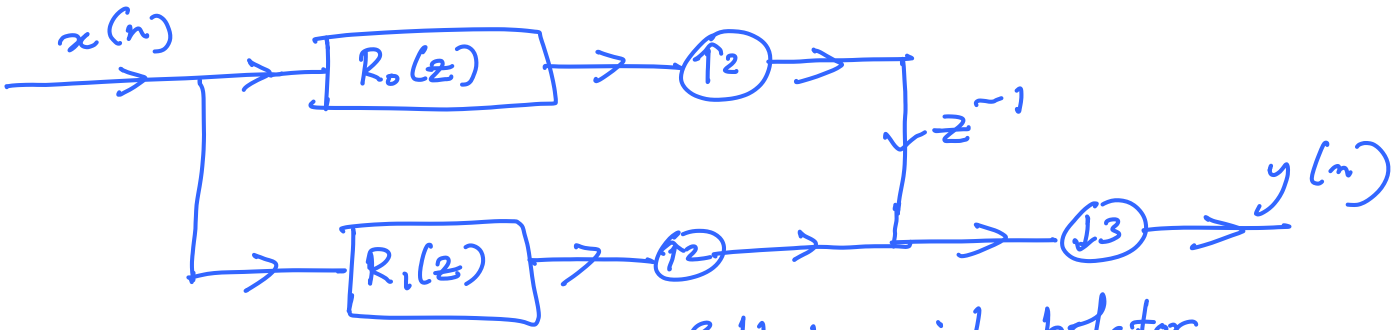
1) At any point in time, $L-1$ out of L multipliers have zero inputs.
2) Only one out of M o/p samples is being retained.

Consider $M=3$ and $L=2$

Using type 1 polyphase decomposition,



Structure (a) : Exploiting the 'decimator'



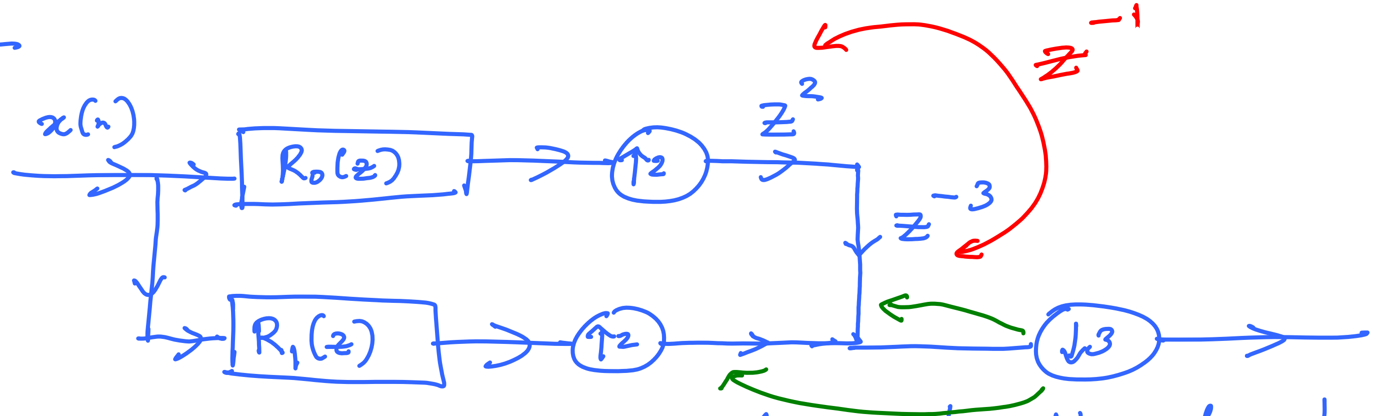
Structure (b) : Exploiting interpolator

Qn : Can we exploit both (a) and (b) to take "full advantage" of decimator & expander ?

We adopt a technique by Hsiao (1990) (Efficient architecture)

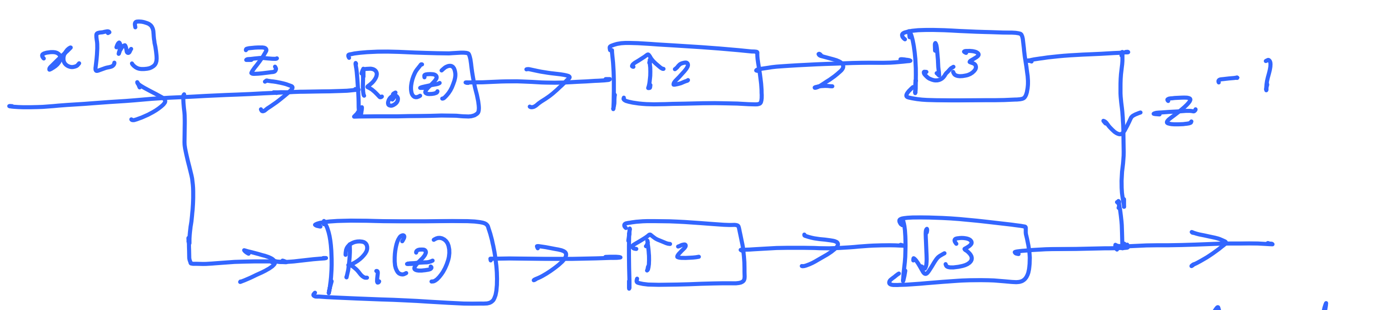
Trick: $z^{-1} = z^{-3} \cdot z^{2}$

(a)

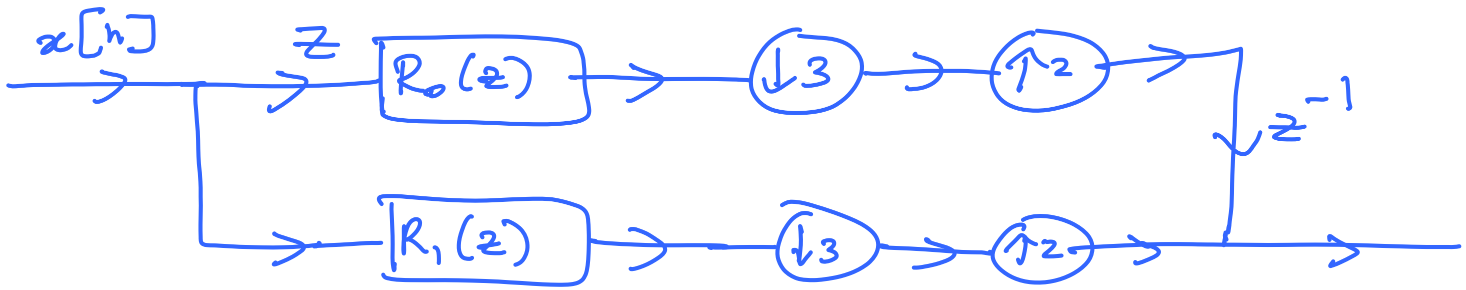


Next, we push the down sampler into the branches before, and apply the following Noble identities.

(b)



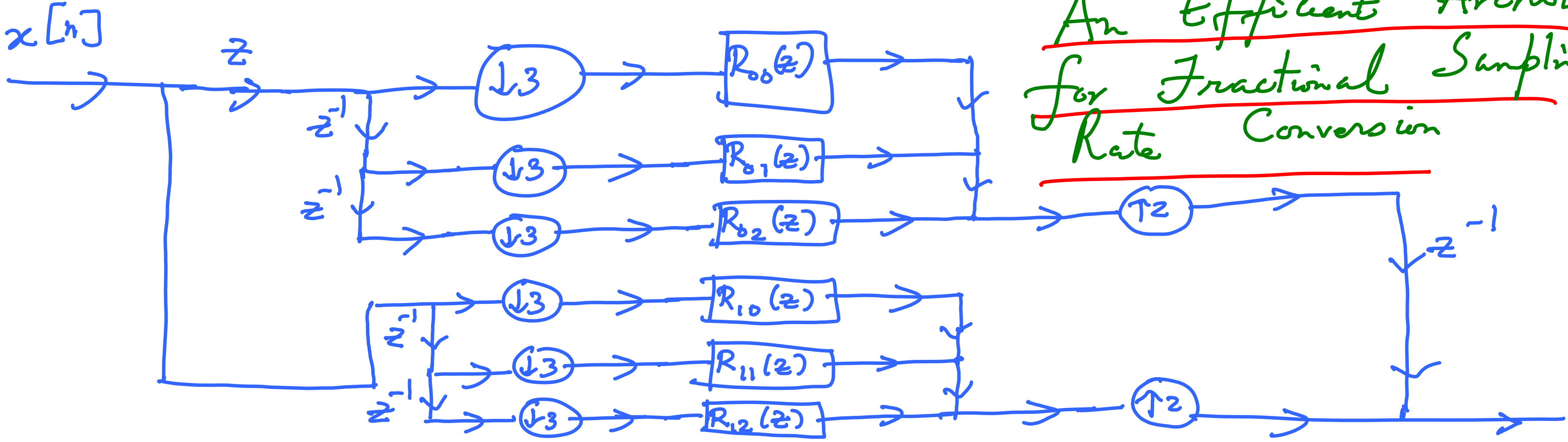
Let us interchange the decimator & the expander since $\gcd(3, 2) = 1$



(c) Let us do a type 1 polyphase decomposition on Components $R_0(z)$ & $R_1(z)$

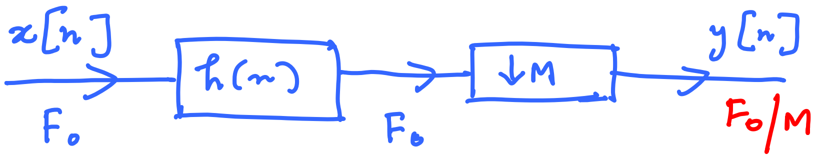
$$R_0(z) = R_{00}(z^3) + z^{-1} R_{01}(z^3) + z^{-2} R_{02}(z^3)$$

$$R_1(z) = R_{10}(z^3) + z^{-1} R_{11}(z^3) + z^{-2} R_{12}(z^3)$$

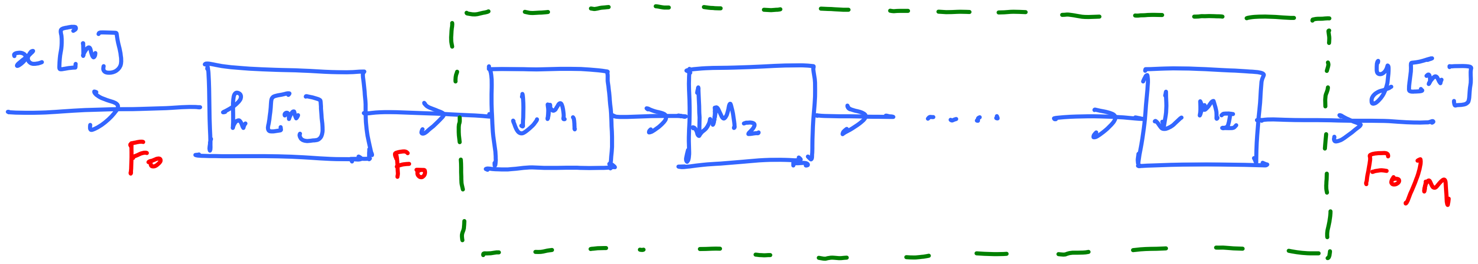


An Efficient Architecture for Fractional Sampling Rate Conversion

Multi Stage Implementations

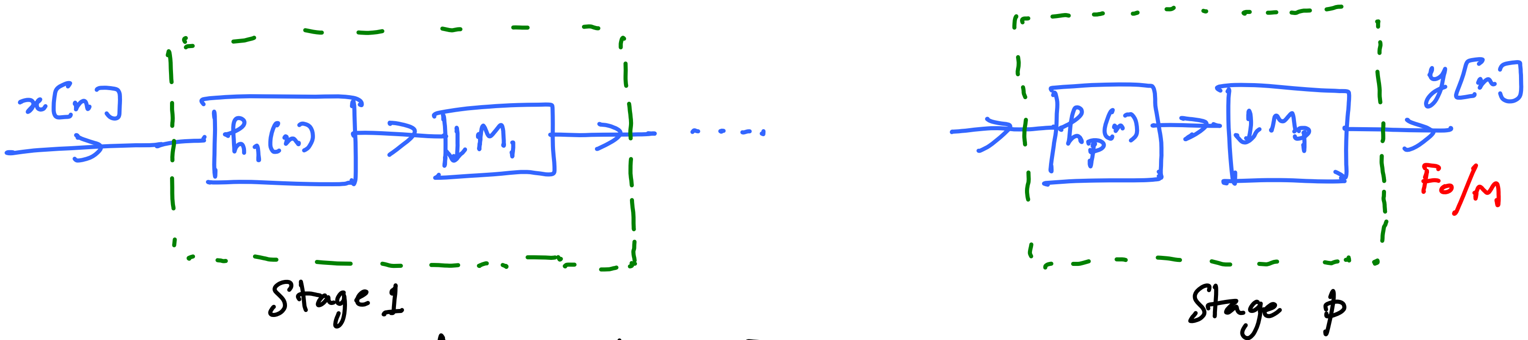


Architecture - I



Architecture - II

$$M = \prod_{i=1}^{I} M_i$$



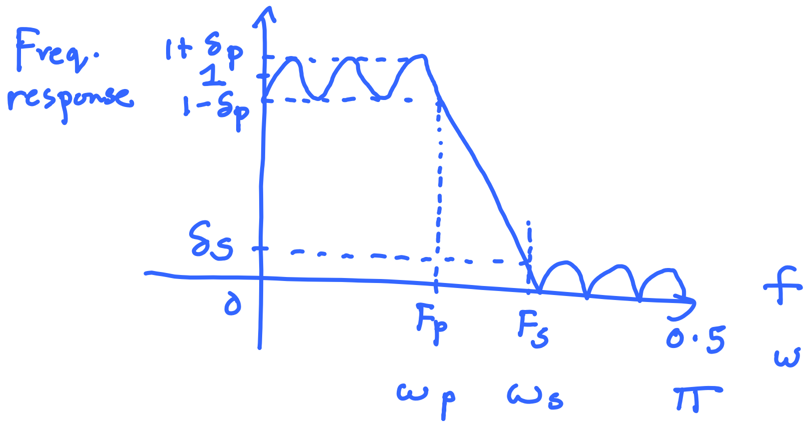
Architecture - III

Question: It seems obvious that we are greatly increasing the computations by having many intermediate filters at various stages. This is opposite to our intention.

Goal: To show why multistage structures are advantageous over a single stage.

Design Example

A signal $x(n)$ with sampling rate 10 KHz is to be down sampled by a factor $M = 100$ to produce a signal at 100 Hz rate. The pass band of the signal is from 0 - 45 Hz and the band from 45 - 50 Hz is the transition band. A pass band ripple of 0.01 and a stop band ripple of 0.001 are desired/required.



$F_s = 50$ Hz

$F_p = 45$ Hz

$\Delta F = 5$ Hz

$\delta_p = 0.01$

$\delta_s = 0.001$

$$N = 1 + \frac{D_\infty(\delta_p, \delta_s)}{\Delta F/F} - f(\delta_p, \delta_s)\, \Delta F/F$$

$$D_\infty(\delta_p, \delta_s) = \left[a_1 (\log_{10} \delta_p)^2 + a_2 \log(\delta_p) + a_3\right] \log(\delta_s) + \left[a_4 (\log_{10} \delta_p)^2 + a_5 \log_{10}(\delta_p) + a_6\right]$$

$$\boxed{\begin{array}{lll} a_1 = 5.3e-3 & a_3 = -0.4761 & a_5 = -0.5941 \\ a_2 = 0.071 & a_4 = -0.0026 & a_6 = -0.4278 \end{array}}$$

Ref: L. R. Rabiner et al.

Some comparisons of FIR and IIR digital filters

Bell. Sys. Tech. Journal, vol. 53, no. 2, Feb. 1974.

$$f(\delta p, \delta s) = 0.512 \log_{10}\left(\frac{\delta p}{\delta s}\right) + 11.01$$

Less accurate but simplified version

$$N = \frac{-10 \log_{10}(\delta p \cdot \delta s) - 15}{14 \Delta F / F} + 1$$

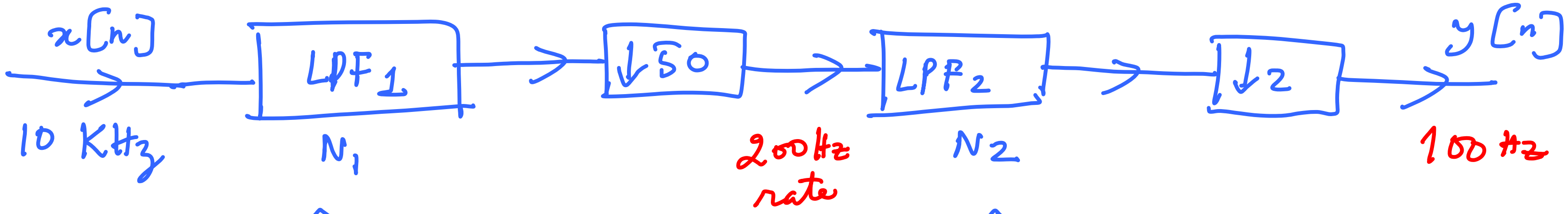
Let us compute all the quantities (For Single Stage design)

$$N \approx \frac{D_\infty(\delta_p, \delta_s)}{\Delta F / F} \approx \frac{D_\infty(0.01, 0.001)}{5 / 10\,\mathrm{KHz}} = 5080$$

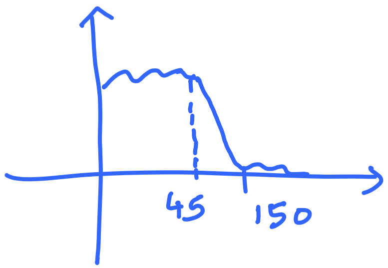
The # of multiplications / sec. needed to implement this would be

$$\frac{NF}{2M} = \frac{5080 \times 10\,\mathrm{KHz}}{2 \times 100} \simeq 2.54\ \frac{\mathrm{Mmul}}{s}.$$

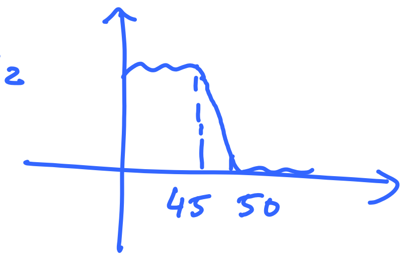
possibly by exploiting filter symmetry



LPF1



LPF2



a) transition band extends from 45 Hz – 150 Hz (Controls filter order)

b) Sampling rate @ o/p is 200 Hz; Nyquist is 100 Hz

$\Rightarrow$ aliasing of frequencies from (100 – 150) Hz to the (50 – 100 Hz) band

Go to Filter 2 i.e., LPF_2 to remove "aliasing" energy

Filter Specifications for 2- stage design

1) Pass band ripple for each stage is approx. $\delta_p/2$
(Pass band ripples add up with a <u>'cascade'</u>)

2) Stop band ripple only gets reduced

<u>1^{st} stage</u>

$$N_1 \approx \frac{D_\infty(\delta_p/2, \delta_s)}{(150-45)/10\,KHz} \approx 263$$

$$\text{Multiplications}/s \text{ for stage } 1 = \frac{NF}{2M_1} = \frac{263 \times 10\,KHz}{2 \times 50} = \frac{52,600}{2} \text{ Mults}/s$$

2nd Stage

$$N_2 = \frac{D_\infty(\delta p/2, \delta_s)}{(50-45)/200} \cong 111$$

$$\text{Multiplications}/s = \frac{111 \times 200}{2 \times 2} = \frac{11\,000}{2}\ \text{Muls}/s.$$

Overall computations for the 2-Stage design

$$\frac{1}{2}(52{,}600 + 11{,}000)\ \text{Muls}/s$$

Assuming filter symmetry

Comparing to the single stage design, we have nearly 8 : 1 improvement (SIGNIFICANT !)

Advantages of multistage designs

1) Significantly reduced computations.
2) Reduced storage
3) Simplified filter design
4) Reduced finite word length effects (lower round off noise & coeff sensitivity)

Drawbacks

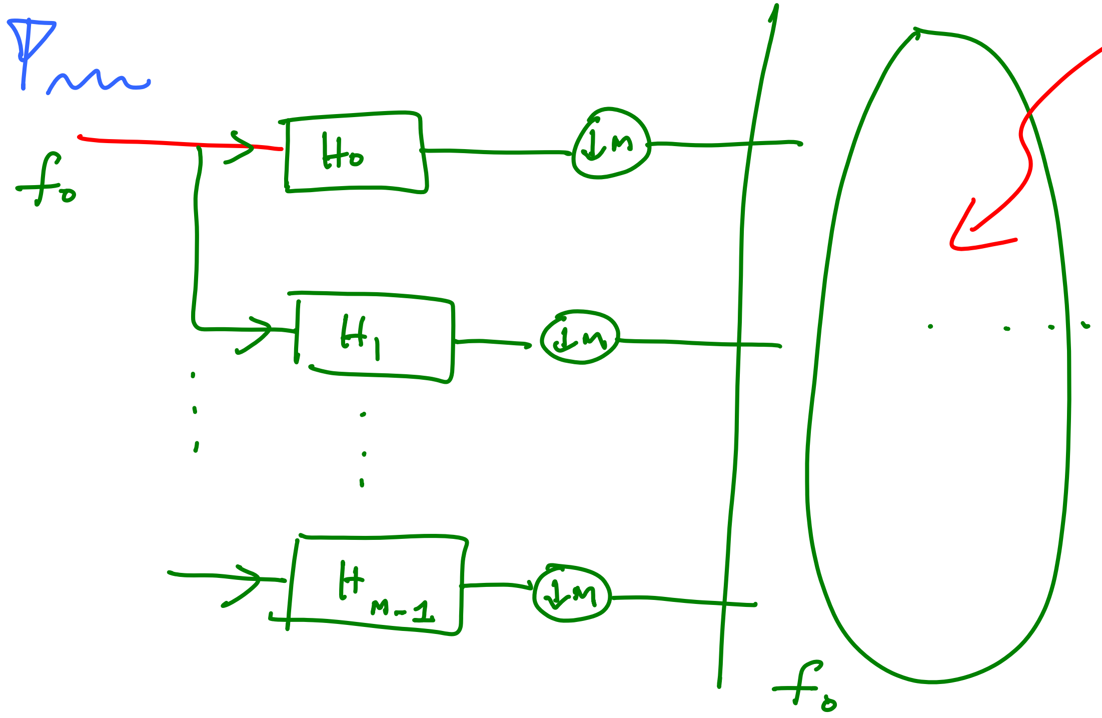
1) Increased control structure required to implement the design

2) Do your design over all choice of M_i's

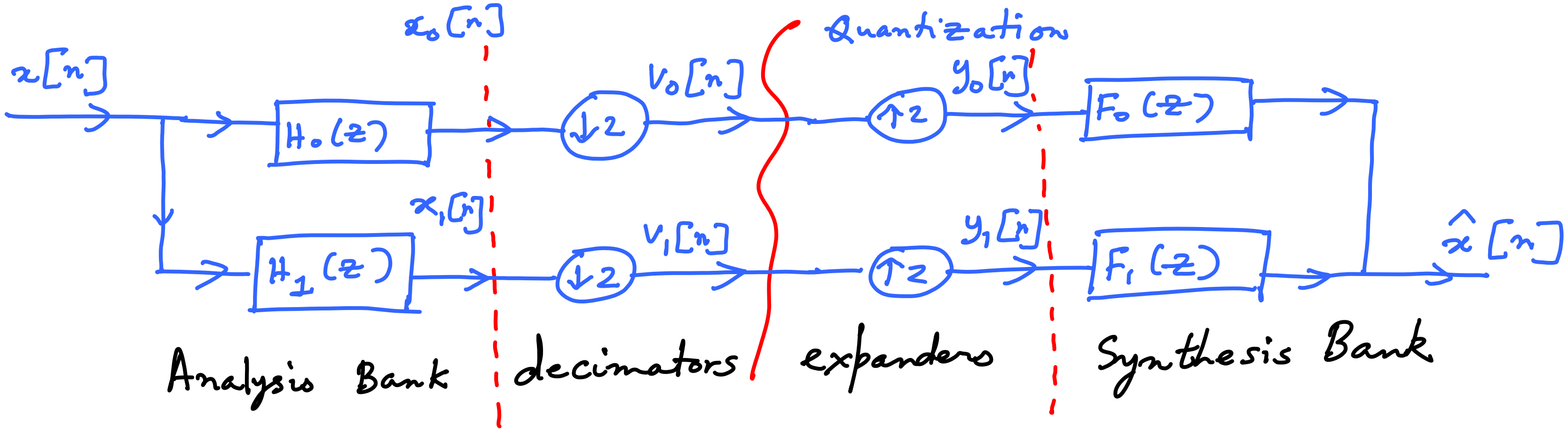
Ex: 100 = 50×2
100 = 10×10
100 = 20×5
100 = 25×4

2 - channel Filter Bank

Filter Bank : Basically a bank of filters



Coding / Compression etc

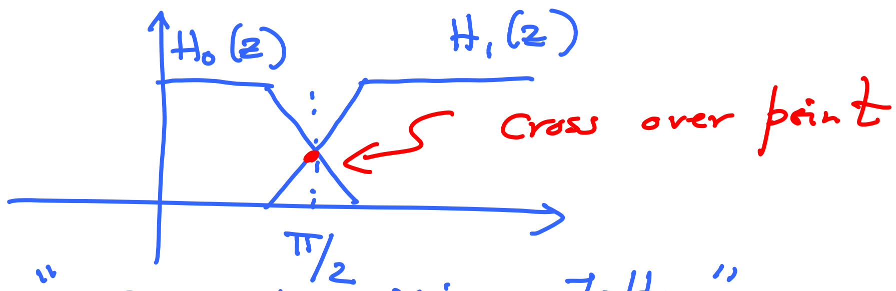


The decimated signals $v_0[n]$ and $v_1[n]$ are coded in such a way that the 'special properties' of the subband are exploited.

These properties include:

a) energy level

b) perceptive importance



"Quadrature Mirror Filters"

The reconstructed signal $\hat{x}[n]$ can suffer from several artifacts

a) Aliasing error

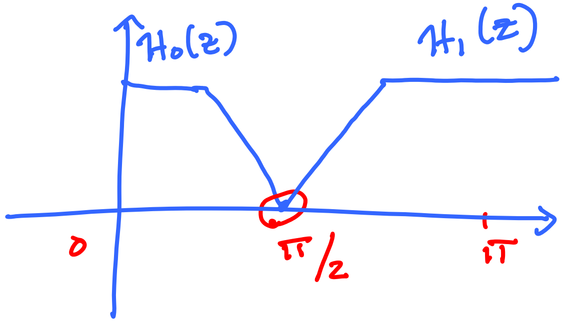
b) Amplitude distortion

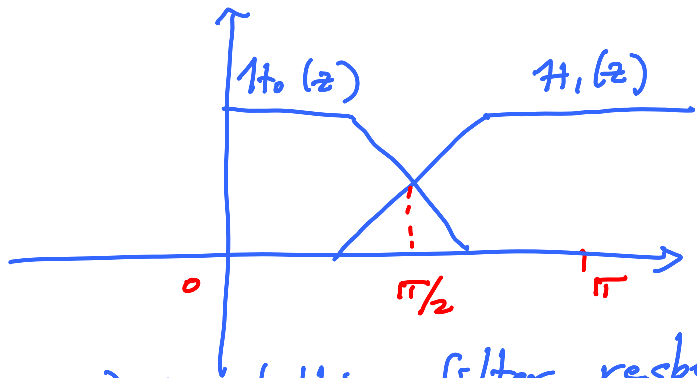
c) Phase distortion

Our goal would be to design synthesis filters to overcome these limitations.

Aliasing / Imaging

In practice, analysis filters have non zero transition bandwidth & stop band gain





a) Stop band attenuations are sufficiently large
"Effect of aliasing" is not predominant

b) Severe attenuation @ $\pi/2$

c) Noise amplification during synthesis.

a) Overlapping filter responses
$\Longrightarrow$ Possibility of aliasing

b) There is gradual attenuation @ $\pi/2$
$\Longrightarrow$ Noise amplification may not be all that bad during synthesis

Expression for the reconstructed signal

$$X_k(z) = H_k(z)\,X(z) \qquad k = 0, 1$$

With $M = 2$, the o/p of the decimators are

$$V_k(z) = \frac{1}{2}\left[X_k(z^{1/2}) + X_k(-z^{1/2})\right]$$

'aliasing part'

Post expansion,

$$Y_k(z) = V_k(z^2)$$

$$= \frac{1}{2}\left[X_k(z) + X_k(-z)\right]$$

$$= \frac{1}{2}\left[H_k(z)X(z) + H_k(-z)X(-z)\right]$$

Reconstructed signal $\hat{X}(z)$ is

$$\hat{X}(z) = F_0(z)\,Y_0(z) + F_1(z)\,Y_1(z)$$

$$\therefore \hat{X}(z) = \frac{1}{2}\Big[H_0(z)\,F_0(z) + H_1(z)\,F_1(z)\Big]X(z) + \underbrace{\frac{1}{2}\Big[H_0(-z)\,F_0(z) + H_1(-z)\,F_1(z)\Big]X(-z)}_{\text{aliasing part}}$$

$$2\hat{X}(z) = \underbrace{\begin{bmatrix} X(z) & X(-z) \end{bmatrix}}_{\text{due to decimation}} \overbrace{\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}}^{\text{Alias component matrix (A C matrix)}} \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}}_{\text{Synthesis filters}}$$

For alias cancelation,

$$H_0(-z)\, F_0(z) + H_1(-z)\, F_1(z) = 0$$

$$\left.\begin{aligned} F_0(z) &= H_1(-z) \\ F_1(z) &= -H_0(-z) \end{aligned}\right\}$$

Given $\{H_0, H_1\}$, we can obtain $\{F_0(z), F_1(z)\}$ as above to remove 'alinsing' error

Analysis bank

NOTE: If we adopt the QMF design, we can construct $H_1(z)$ from $H_0(z)$ $\quad H_1(z) = H_0(-z)$

With QMF, given H_0, we can construct & populate all the other filters towards alias cancelation.

Amplitude & Phase distortion

With a 2 channel filter bank, free of aliasing,

$\hat{X}(z) = T(z)\, X(z)$ where

$$T(z) = \underbrace{\frac{1}{2}\left[H_0(z)\, F_0(z) + H_1(z)\, F_1(z) \right]}_{\text{distortion transfer function}}$$

$$T(z) = \frac{1}{2}\left[H_0(z)\, H_1(-z) - H_1(z)\, H_0(-z) \right]$$

$$T(z)\Big|_{z=e^{j\omega}} = \left| T(e^{j\omega}) \right| e^{j\phi(\omega)}$$

Unless $|T(e^{j\omega})| = d \neq 0$ $\forall \omega$, we have magnitude distortion

Unless $\phi(\omega) = a + b\omega$ $\hat{X}(e^{j\omega})$ suffers from phase distortion.

Let $V(z) = H_0(z)\, H_1(-z)$

$V(-z) = H_0(-z)\, H_1(z)$

$$T(z) = \frac{1}{2}\left[\, V(z) - V(-z) \,\right]$$

$\Longrightarrow$ $T(z)$ has only odd powers of z

$$T(z) = z^{-1}\, S(z^2)$$

$|T(z)|$ has a period of π instead of 2π !

Perfect Reconstruction Filter Bank

For the PR, $T(z) = c\, z^{-n_0}$

$\Longrightarrow \hat{x}(n) = c\, x(n-n_0)$

Consider the QMF bank system,

Suppose $H_1(z) = H_0(-z) \Longrightarrow$ $H_1(z)$ is a good HPF if $H_0(z)$ is a good LPF!

$$|H_1(e^{j\omega})| = |H_0(e^{j(\pi-\omega)})|$$

$$T(z) = \frac{1}{2}\left[H_0^2(z) - H_1^2(z) \right]$$

$$= \frac{1}{2}\left[H_0^2(z) - H_0^2(-z) \right]$$

For phase distortion :

Let $H_0(z) = \sum_{n=0}^{N} h_0(n) z^{-n}$ $\quad h_0(n)$ is real

Let $h_0(n) = \pm h_0(N-n)$ (linear phase)

But for LPF, $h_0(n) = h_0(N-n)$

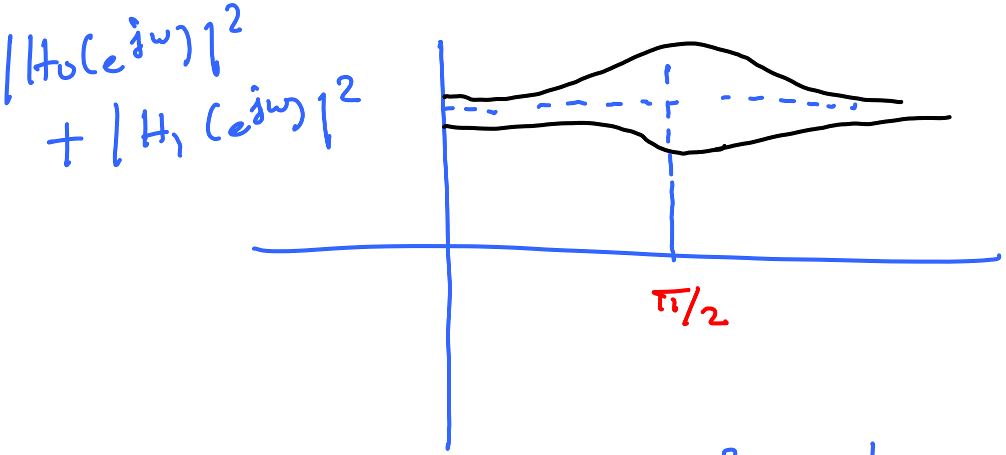
$$H_0(e^{j\omega}) = e^{-j\omega L(\omega)} R(\omega)$$

Exercise!

$$T(e^{j\omega}) = \frac{1}{2} e^{-j\omega N} \left(\left| H_0(e^{j\omega}) \right|^2 - (-1)^N \left| H_0(e^{j(\pi-\omega)}) \right|^2 \right)$$

If N is even, $T(e^{j\omega})$ reduces to zero @ $\omega = \pi/2$ leading to 'severe attenuation'

Minimizing residual amplitude distortion



GOAL: $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$ (approximately)

Let us formulate an objective function

$$\phi = \alpha \, \phi_1 + (1-\alpha) \, \phi_2 \qquad \text{with} \quad 0 < \alpha < 1$$

$$\phi_1 = \int_{\omega_s}^{\pi} \left| H_0(e^{j\omega}) \right|^2 d\omega$$

$$\phi_2 = \int_{0}^{\pi} \left(1 - \left| H_0(e^{j\omega}) \right|^2 - \left| H_0(e^{j(\pi-\omega)}) \right|^2 \right)^2 d\omega$$

$$h_0[n] = \min_{h_0[n]} \phi \qquad \text{(Johnston 1980)}$$