Theorem: If $\{\underline{P}_1, \underline{P}_2, \dots, \underline{P}_m\}$ are mutually orthogonal, they must be linearly independent.

Proof: Suppose they are '<u>linearly dependent</u>'

There are a set of coeffts. $a_1, a_2, \dots a_m$ <u>not all zero</u>

$$\sum_{i=1}^{m} a_i \underline{P}_i = \underline{0} \qquad \text{--------------} \qquad (1)$$

Now, we take the I.P. of (1) over each $\underline{P}_i$.

$$\left\langle \sum_{i=1}^{m} a_i \underline{P}_i , \underline{P}_1 \right\rangle = \langle \underline{0}, \underline{P}_1 \rangle = 0.$$

$$a_1 \langle \underline{P}_1, \underline{P}_1 \rangle = 0.$$

$|||^{ly}$ $\qquad a_2 \langle \underline{P}_2, \underline{P}_2 \rangle = 0$

$$\vdots$$

$$a_m \langle \underline{P}_m, P_m \rangle = 0$$

Now $\{\underline{P}_i\}_{i=1}^m \neq \underline{0}$ i.e., each of $\{\underline{P}_i\}_{i=1}^m$

are 'non zero',

$\Longrightarrow \langle \underline{P}_i, \underline{P}_i \rangle \neq 0 \Longrightarrow a_1 = a_2 = \cdots = a_m = 0$

$\Longrightarrow \{\underline{P}_i\}_{i=1}^m$ are linearly independent; contradiction

NOTE: $\{(0,1),(1,0)\}$ Mutually orthogonal $\Longrightarrow$ Linearly independence $\boxtimes$

$\{(1,1),(1,0)\}$ $\qquad \not\Longleftarrow$

Weighted Inner Products

$$\langle \underline{x}, \underline{y} \rangle_W = \underline{y}^T W \underline{x}$$

Can $\langle \underline{x}, \underline{y} \rangle_W$ be used as a norm ?

It cannot be used as a norm ; Counter : $\underline{x}^T W \underline{x} > 0$

$x \neq 0 \ \forall \underline{x}$

Exercise : $\underline{x} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$ $\quad \underline{x}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} x > 0$

$\forall \underline{x}, x \neq 0$

$$\begin{bmatrix} \alpha & \alpha \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha^2 (a+b+c+d); -$$

If $a+b+c+d = 0$ or in various comb. of a, b, c, d to sum to 0

Irrespective of $\underline{x} \neq \underline{0}$ $\langle \underline{x}, \underline{x} \rangle_W = 0$

Expectations as an innerproduct

$$E(x^2) = \langle x, x \rangle$$

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x\, y \underbrace{f_{XY}(x, y)}_{\text{joint density of 2 RVs}} \, dx \, dy$$

$$\langle x+z, y \rangle = \int_{-\infty}^{\infty} (x+z) \cdot y \, f_{XYZ}(x, y, z) \, dx \, dy \, dz$$

$$E(x^2) = 0 \Rightarrow \int_{-\infty}^{\infty} x^2 f_X(x) \, dx$$

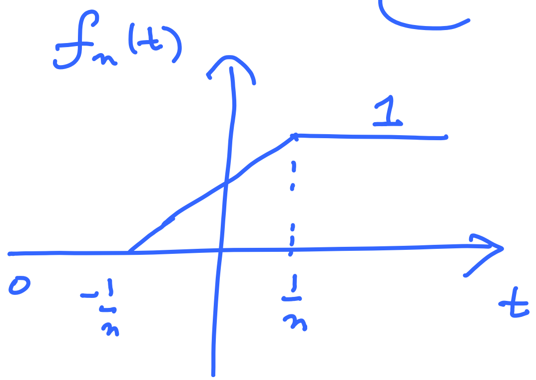
Hilbert & Banach spaces

Defn: (1) A complete normed V.S is a Banach space

(2) A complete normed V.S. with an Inner Product (i.e., norm is the induced norm) is called a Hilbert space.

Example:
Space of all continuous functions 'C' over [a, b] forms a Banach space under L_∞ but not for L_p $(p < \infty)$ as some sequence of functions may not have a limit.

$$f_n(t) = \begin{cases} 0 & t < -\frac{1}{n} \\ nt/2 + 1/2 & -1/n \le t \le \frac{1}{n} \\ 1 & t > \frac{1}{n} \end{cases}$$



$$\lim_{n \to \infty} f_n(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

Limiting fn 'discontinuous' $\notin C$

Orthogonal Subspaces

Defn: Let S be a V.S. Let V and W be subspaces of S. V and W are 'orthogonal' if every vector $\underline{v} \in V$ is 'orthogonal' to every vector $\underline{w} \in W$ i.e., $\langle \underline{v}, \underline{w} \rangle = 0$

Defn: For a subset V of an I.P. space S, the space of all vectors orthogonal to V is called the orthogonal complement denoted by $V^{\perp}$

Linear Transformations

Defn: $L : X \to Y$ over a same scalar field $\mathbb{R}$.

is a linear transformation if the foll. hold.
(transformation)

$L(\alpha \underline{x})$ for some $\underline{x} \in X$ $= \alpha L(\underline{x})$ (a)

$L(\underline{x}_1 + \underline{x}_2) = L(\underline{x}_1) + L(\underline{x}_2)$ (b)

Using (a) & (b)

$$L(\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2) = \alpha_1 L(\underline{x}_1) + \alpha_2 L(\underline{x}_2)$$

Exercise : If X is the set of Fourier transformable functions
Let Y be the set of Fourier transforms of elements in X

$$F : X \to Y$$

$$F\left(x(t)\right) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Check if 'F' is a linear operator.

Defn:

Range space of L = $R(L) = \{ \underline{y} = L\underline{x} : x \in X \}$

Null space of L = $N(L) = \{ \underline{x} \in X : L\underline{x} = \underline{0} \}$

Null space of an operator is called the 'kernel' of the operator.

Let $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ be a basis for an innerproduct space V.

<u>Theorem</u> : If the basis $\underline{v}_1\ \underline{v}_2 \dots \underline{v}_n$ is an <u>orthogonal</u> set, then for any $\underline{x} \in V$.

$$\underline{x} = \frac{\langle \underline{x}, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1 + \frac{\langle \underline{x}, \underline{v}_2 \rangle}{\langle \underline{v}_2, \underline{v}_2 \rangle} \underline{v}_2 + \dots + \frac{\langle \underline{x}, \underline{v}_n \rangle}{\langle \underline{v}_n, \underline{v}_n \rangle} \underline{v}_n$$

If $\underline{v}_1, \underline{v}_2 \dots \underline{v}_n$ are orthonormal,

$$\underline{x} = \langle \underline{x}, \underline{v}_1 \rangle \underline{v}_1 + \langle \underline{x}, \underline{v}_2 \rangle \underline{v}_2 + \dots + \langle \underline{x}, \underline{v}_n \rangle \underline{v}_n$$

Proof :

$$\underline{x} = x_1 \underline{v}_1 + \cdots + x_n \underline{v}_n$$ ($\because$ Basis, linearly independent)

$$\langle \underline{x}, \underline{v}_i \rangle = x_i \langle \underline{v}_i, \underline{v}_i \rangle$$ ($\because$ orthogonality)

$$\underline{x} = \sum_{i=1}^{n} \frac{\langle \underline{x}, \underline{v}_i \rangle}{\langle \underline{v}_i, \underline{v}_i \rangle} \underline{v}_i$$

$$\underline{x} = \sum_{i=1}^{n} \langle \underline{x}, \underline{v}_i \rangle \underline{v}_i$$ ($\because$ orthonormal $\langle \underline{v}_i, \underline{v}_i \rangle = \underline{1}$)

$\boxtimes$

Gram Schmidt Orthogonalization

<u>Motivation</u> : Construction of an orthogonal basis for a vector space <u>OR</u> an orthogonal basis for a signal space

Suppose we are given $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n$

Let $\underline{v}_1 = \underline{x}_1$

$$\underline{v}_2 = \underline{x}_2 - \frac{\langle \underline{x}_2, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1$$

$$\underline{v}_3 = \underline{x}_3 - \frac{\langle \underline{x}_3, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1 - \frac{\langle \underline{x}_3, \underline{v}_2 \rangle}{\langle \underline{v}_2, \underline{v}_2 \rangle} \underline{v}_2$$

$$\vdots$$

$$\underline{v}_n = \underline{x}_n - \sum_{i=1}^{n-1} \frac{\langle \underline{x}_n, \underline{v}_i \rangle}{\langle \underline{v}_i, \underline{v}_i \rangle} \underline{v}_i$$

<u>Claim:</u> The set $\{\underline{v}_i\}_{i=1}^{n}$ forms an orthogonal basis for V.

If we normalize $\dfrac{\underline{v}_i}{\|\underline{v}_i\|}$, it forms an orthonormal set

<u>Exercise:</u> Prove this claim by any of your favourite methods

Example : Consider the foll. vectors in $\mathbb{R}^2$ (Numerical ex.)

$$S = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$$

$$\underline{v}_1 = \underline{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\underline{v}_2 = \underline{x}_2 - \frac{\langle \underline{x}_2, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1$$

$$= \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \frac{14}{13} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Verify $\langle \underline{v}_1, \underline{v}_2 \rangle = 0$

$$\begin{bmatrix} 10/13 \\ -15/13 \end{bmatrix}$$

$$\underline{e}_1 = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \underline{e}_2 = \frac{1}{\sqrt{\left(\frac{10}{13}\right)^2 + \left(\frac{15}{13}\right)^2}}$$

Ex: Suppose we have $\cos(t)$ & $\sin(t)$ defined over $[0, 2\pi]$

$$f(t) = t^{3/2}$$

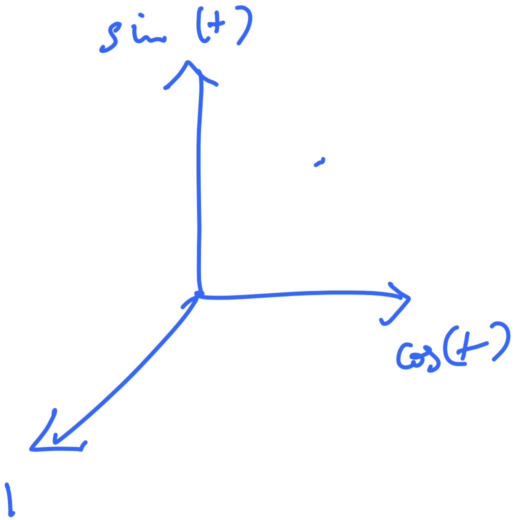
What we need is,

$$f(t) \approx a_0 \cdot 1 + a_1 \frac{\cos(t)}{\|\cdot\|} + a_2 \frac{\sin(t)}{\|\cdot\|}$$

$$\int_0^{2\pi} 1 \cdot \cos(t)\, dt = 0$$

$$\int_0^{2\pi} 1 \cdot \sin(t)\, dt = 0$$

$$\int_0^{2\pi} \sin(t) \cos(t)\, dt = 0$$



$C[0, 2\pi]$

3-dim space

Linear Approximation in a Signal Space

We would like a signal $s(t)$ to be a weighted linear sum of $\{f_k(t)\}_{k=1}^{N}$

$$\hat{s}(t) = \sum_{k=1}^{N} s_k f_k(t) \qquad \text{——— (1)}$$

Consider $$\mathcal{E} = \int_{-\infty}^{\infty} \left(s(t) - \hat{s}(t) \right)^2 dt \qquad \text{——— (2)}$$

Plug (1) in (2)

$$\mathcal{E} = \int_{-\infty}^{\infty} \left(s(t) - \sum_{k=1}^{N} s_k f_k(t) \right)^2 dt$$

$$\frac{\partial \mathcal{E}}{\partial s_n} = -2 \int_{-\infty}^{\infty} \left(s(t) - \sum_{k=1}^{N} s_k f_k(t) \right) f_n(t)\, dt = 0$$

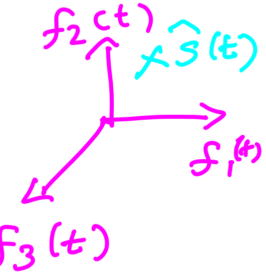
$$\Rightarrow \int_{-\infty}^{\infty} s(t) f_n(t)\, dt = \int_{-\infty}^{\infty} s_n f_n^2(t)\, dt$$

$$\left(\because \text{Orthogonality} \right)$$

$$\int_{-\infty}^{\infty} s(t) f_n(t)\, dt = s_n \underbrace{\int_{-\infty}^{\infty} f_n^2(t)\, dt}_{1} \quad \left(\because \text{Ortho normality} \right)$$

$$s_n = \langle s(t), f_n(t) \rangle$$

We need to compute the squared error



$$\mathcal{E} = \underbrace{\int_{-\infty}^{\infty} s^2(t)\, dt}_{E_s} - \underbrace{2 \int_{-\infty}^{\infty} s(t) \sum_{k=1}^{N} s_k f_k(t)\, dt}_{2\sum_{k=1}^{N} s_k s_k}$$

$$+ \int_{-\infty}^{\infty} \sum_{k=1}^{N} s_k f_k(t) \sum_{l=1}^{N} s_l f_l(t)\, dt$$

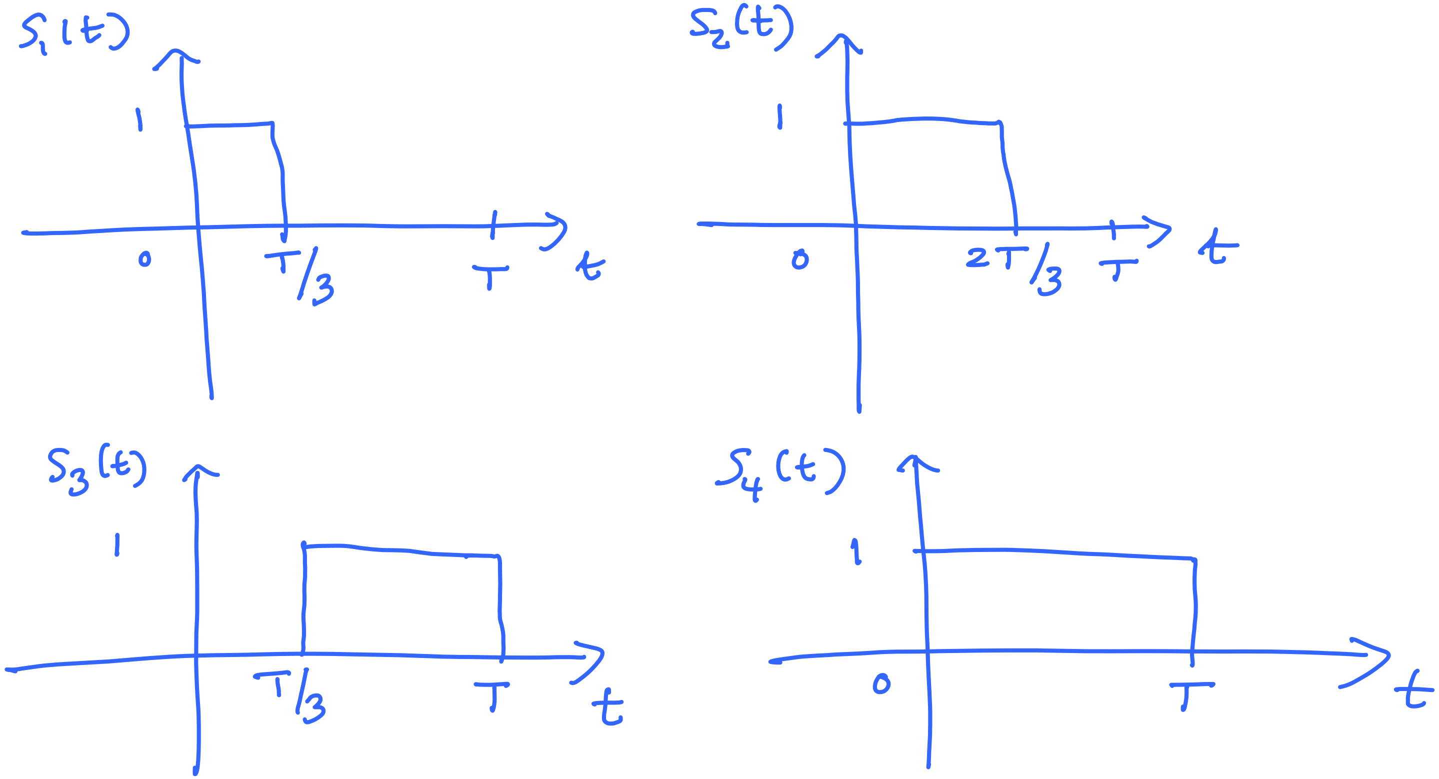
$$\mathcal{E} = E_s - 2\sum_{k=1}^{N} s_k s_k + \underbrace{\sum_{k=1}^{N} \sum_{l=1}^{N} s_k s_l \int_{-\infty}^{\infty} f_k(t) f_l(t)\, dt}_{+\sum_{k=1}^{N} s_k^2}$$

$$\mathcal{E} = \underbrace{E_s}_{} - \underbrace{\sum_{k=1}^{N} s_k^2}_{}$$

(Square norm of $\hat{s}(t)$ in the signal space)

Gram Schmidt Orthogonalization for signals

Example :



$$s_4(t) = s_1(t) + s_3(t)$$ (Linearly dependent)

Qn: Can I construct an orthonormal set of basis for these signals?

Let us apply G. S. O. for signals

$$f_1(t) = \frac{s_1(t)}{\|s_1(t)\|} = \frac{s_1(t)}{\sqrt{\int_0^T s_1^2(t)\,dt}} = \begin{cases} \sqrt{\frac{3}{T}} & 0 \le t \le \frac{T}{3} \\ 0 & \text{else} \end{cases}$$

$|||^{ly}$

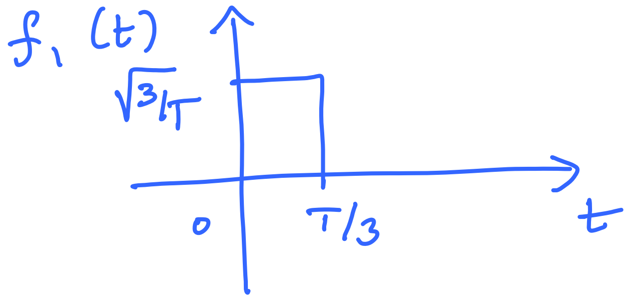
$$f_2(t) = \frac{s_2(t) - s_{21} f_1(t)}{\|s_2(t) - s_{21} f_1(t)\|} = \begin{cases} \sqrt{\frac{3}{T}} & \frac{T}{3} \le t \le \frac{2T}{3} \\ 0 & \text{else} \end{cases}$$

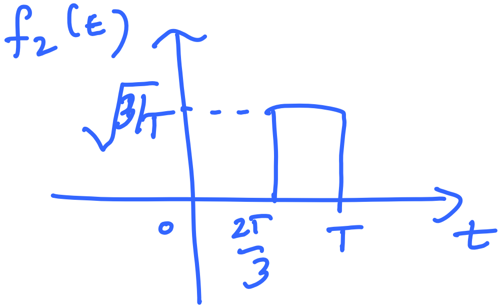
$$s_{ij} \triangleq \int_0^T s_i(t) f_j(t)\,dt$$

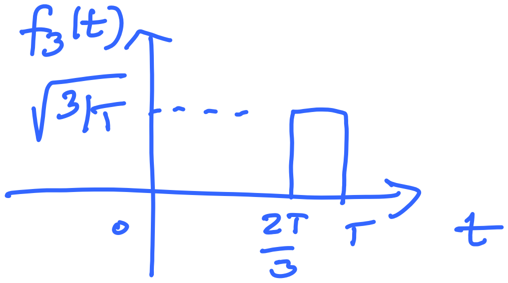
$|||^{ly}$

$$f_3(t) = \begin{cases} \sqrt{\frac{3}{T}} & \frac{2T}{3} \le t \le T \\ 0 & \text{else} \end{cases}$$

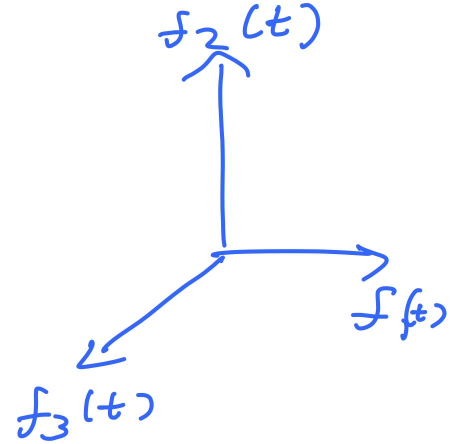
$$s_{21} = \int_0^T s_2(t) f_1(t)\,dt$$



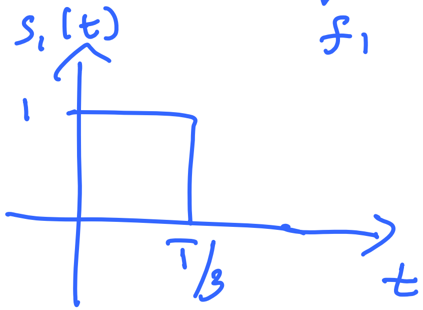




$$\underline{s}_1 = \Big(\underbrace{\sqrt{T/3}}_{f_1},\ \underbrace{0}_{f_2},\ \underbrace{0}_{f_3}\Big)$$



$\underline{s}_1 \equiv$



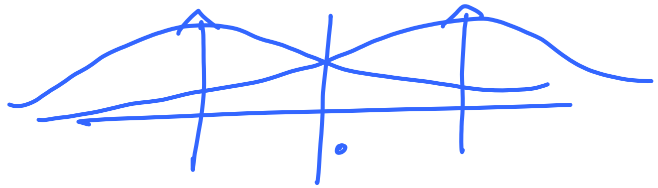
$$\underline{s}_2 = \left(\sqrt{T/3},\ \sqrt{T/3},\ 0\right)$$

$$\underline{s}_3 = \left(0,\ \sqrt{T/3},\ \sqrt{T/3}\right)$$

$$\underline{s}_4 = \left(\sqrt{T/3},\ \sqrt{T/3},\ \sqrt{T/3}\right)$$

They reside in a 3-dim. signal space spanned by $f_1(t)$, $f_2(t)$ & $f_3(t)$

$\underline{s}_1$ $\underline{s}_2$ $\underline{s}_3$ s_4



$$\langle \underline{s}_3 , \underline{s}_4 \rangle = \int_0^T s_3(t)\, s_4(t)\, dt$$

$\langle \underline{s}_3 , \underline{s}_3 \rangle$

$\langle \underline{s}_4 , \underline{s}_4 \rangle$

$$\cos\theta = \frac{\langle \underline{s}_3 , \underline{s}_4 \rangle}{\sqrt{\langle s_3 , s_3 \rangle}\ \sqrt{\langle s_4 , s_4 \rangle}}$$

$\sin(\omega t)$ $\sin(\omega t + \theta)$