Problem 8.5.

Solution. For the matched filter considered in Example 2, it is given that the random vector $X = s + V$, where the signal component is represented by $s$ has a fixed Euclidean norm of one, i.e., $\sqrt{s^T s} = 1 \implies s^T s = 1$.

The random vector $V$, represents the additive noise component and has zero mean and covariance matrix $\sigma^2 I$. The correlation matrix of $X$ is given by $R = ss^T + \sigma^2 I$.

The largest eigenvalue of the correlation matrix $R$ and the corresponding eigenvector are $\lambda_1 = 1 + \sigma^2$ and $q_1 = s$ respectively. Let us post multiply the correlation matrix by the eigenvector $q_1$.

\[
Rq_1 = (ss^T + \sigma^2 I) q_1 \\
= (ss^T + \sigma^2 I) s \\
= ss^T s + \sigma^2 s \\
= s + \sigma^2 s \quad \text{(since } s^T s = 1) \\
= (1 + \sigma^2) s \\
= \lambda_1 s \\
= \lambda_1 q_1
\]

Therefore the given parameters satisfy the basic relation of $Rq_1 = \lambda_1 q_1$. ■

Problem 8.15.

Solution. Let the total number of inputs be $N$. The data $x_i$ is projected using a kernel to get the projected data set $\phi(x_i)$. Let $\tilde{\phi}(x_i)$ be the projected data points after centralizing the data as given below

\[
\tilde{\phi}(x_i) = \phi(x_i) - \frac{1}{N} \sum_{i=1}^{N} \phi(x_i).
\]
We can get the corresponding elements of the gram matrix $\tilde{K}$ as below:

$$
\tilde{k}_{ij} = \tilde{\phi}^T(x_i)\tilde{\phi}(x_j)
$$

$$
= \left( \phi(x_i) - \frac{1}{N} \sum_{m=1}^{N} \phi(x_m) \right)^T \left( \phi(x_j) - \frac{1}{N} \sum_{n=1}^{N} \phi(x_n) \right)
$$

$$
= \phi(x_i)^T \phi(x_j) - \frac{1}{N} \sum_{m=1}^{N} \phi^T(x_m) \phi(x_j) - \frac{1}{N} \sum_{n=1}^{N} \phi^T(x_i) \phi(x_n) + \frac{1}{N^2} \sum_{m=1}^{N} \sum_{n=1}^{N} \phi^T(x_m) \phi(x_n)
$$

$$
= k_{ij} - \frac{1}{N} \sum_{m=1}^{N} k_{mj} - \frac{1}{N} \sum_{n=1}^{N} k_{in} - \frac{1}{N^2} \sum_{m=1}^{N} \sum_{n=1}^{N} k_{mn}
$$

A compact representation of the above in the matrix form is given by

$$
\tilde{K} = K - NK - KN + NKN
$$

where $N$ is an $N \times N$ matrix with all entries as $\frac{1}{N}$.

Problem 8.16.

Solution. It is given that the eigenvector $\tilde{q}$ of the correlation matrix $\tilde{R}$ is normalized to unit length, that is,

$$
\tilde{q}_k^T \tilde{q}_k = 1 \quad \text{for } k = 1, 2, \ldots, l
$$

where it is assumed that the eigenvalues of $K$ are arranged in descending order with $\lambda_l$ being the smallest nonzero eigenvalue of the Gram matrix $K$. We know that

$$
\tilde{q} = \sum_{j=1}^{N} \alpha_j \phi(x_j)
$$

$$
K\alpha = \lambda \alpha
$$

Using (2) in (1) and upon simplifying, we get

$$
1 = \tilde{q}_k^T \tilde{q}_k
$$

$$
= \sum_{i=1}^{N} \alpha_i^T \alpha_k \phi(x_i)^T \phi(x_j)
$$

$$
= \alpha_k^T K \alpha_k
$$

$$
= \alpha_k^T \lambda \alpha_k \quad \text{(Using (3))}
$$

$$
= \lambda \alpha_k^T \alpha_k
$$

$$
\frac{1}{\lambda_k} = \alpha_k^T \alpha_k \quad \text{for } k = 1, 2, \ldots, l.
$$

Therefore, we see that normalization of the eigenvector $\alpha$ of the Gram matrix $K$ is equivalent to the requirement of Eq(8.109) to be satisfied.