Homework #3 solutions

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Linear and non-linear programming-1

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Problem 1.

Solution. Let us consider the LPP

$$\begin{aligned} \text{maximize} \quad & 3p_1 + 6p_3 \\ \text{subject to} \quad & 2p_1 + 3p_2 - p_3 \geq 1 \\ & 3p_1 + p_2 - p_3 \leq -1 \\ & -p_1 + 4p_2 + 2p_3 \leq 0 \\ & 3p_1 + p_2 - p_3 \leq -1 \\ & p_1 - 2p_2 + p_3 = 0 \\ & p_1 \leq 0 \\ & p_2 \geq 0 \\ & p_3 \text{ is free} \end{aligned}$$

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Problem 3.

Solution.

(a) False: If the dual basic feasible solution associated with $x^\star$ is infeasible, then the optimal cost is $-\infty$.

(b) True: Phase I is always feasible

(c) True: Let p_i be the free variable corresponding to the i^{th} equality constraint. Removal of i^{th} equality constraint results in absence of p_i. The objective function of the dual is

$$p_1 b_1 + \cdots + p_{i-1} b_{i-1} + p_{i+1} b_{i+1} + \cdots + p_m b_m \tag{1}$$

which is same as the objective function with $p_i = 0$.

(d) True: follows directly from weak duality theorem.

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Problem 4.

Solution.

- $\min_{x\in\Re^n} \max_{i=1,\dots,m} \left(p_i a_i^T x - p_i b_i\right) = p_i v$. Using the given data, we get

$$\min_{x\in\Re^n} \max_{i=1,\dots,m} (-p_i b_i) = p_i v \tag{2}$$

$$\max_{i=1,\dots,m} (-p_i b_i) = p_i v \tag{3}$$

But we know that $0 \le p_i \le 1$ using the upper bound we get

$$-p^T b \le v \tag{4}$$

- Write the dual of the given problem and use strong duality theorem to show that the optimal cost is v.

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Problem 5.

Solution.

1. Assume that **(a)** is true. Then we have $p^T A x \ge 0$. But we know that $Ax = 0$ this results in $P^T = 0^T$. Therefore, **(b)** is false.

2. Assume that **(a)** is false. Then consider the following maximization problem

$$\begin{aligned} \text{maximize} \quad & 0^T x \\ \text{subject to} \quad & Ax = 0 \\ & x \ge 0 \end{aligned}$$

which is infeasible. Therefore, from Farka's lemma we know that $\exists\, p$ such that $p^T A > 0^T$.

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Problem 6.

Solution. The proof has been discussed in class. Please refer to class notes. ■

Problem 7.

Solution.

(a) Let x be optimal point, d be the feasible direction and $\theta > 0$. Define $y = x + \theta d$. We know that $c^{\mathrm{T}} \leq c^{\mathrm{T}} y$. This shows $c^{\mathrm{T}} d \geq 0$. Now consider $c^{\mathrm{T}} d \geq 0$. We know that $d = \frac{1}{\theta}(y - x)$. Therefore, $c^{\mathrm{T}} d$ will result in $c^{\mathrm{T}} y \geq c^{\mathrm{T}} x$. Therefore, x is optimal.

(b) Let d be a non-zero feasible direction and let x be unique optimal point. We have $c^{\mathrm{T}} x < c^{\mathrm{T}}(x + \theta d)$ which results in $c^{\mathrm{T}} d > 0$. Let $c^{\mathrm{T}} d > 0$. Define $d = \frac{1}{\theta}(y - x)$. We see that $c^{\mathrm{T}} \frac{1}{\theta}(y - x) > 0$ results in $c^{\mathrm{T}} y > c^{\mathrm{T}} x$.

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Problem 8.

Solution. Consider a point $x \in P$. Let $\theta > 0$ and let $y = x + \theta d$. For d to be a feasible direction, we need $Ay = b$ and $y \geq 0$. It is easy to see that d is feasible iff $Ad = 0$. Also, $y \geq 0 \implies x + \theta d \geq 0$. Now, with $x_i = 0$ we see that $d_i \geq 0$. ■

Problem 9.

Solution. The set P is characterized by the following conditions:

1. $x_1 + x_2 + x_3 = 1$
2. $x \geq 0$

Let $y = x + \theta d$, with $x = (0, 0, 1)$ we have $y = (\theta d_1, \theta d_2, 1 + \theta d_3)$. For $y \in P$, we require

$$(d_1 + d_2 + d_3) = 0 \tag{5}$$

and

$$\begin{aligned} d_1 &\geq 0, & (6) \\ d_2 &\geq 0, & (7) \\ 1 + \theta d_3 &\geq 0 & (8) \end{aligned}$$

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From (5) and (8), we have

$$d_3 = -d_1 - d_2 \tag{9}$$

Combining (6), (7) and (9) in (8) we get

$$\theta \leq \frac{1}{d_1 + d_2} \tag{10}$$

Therefore, feasible direction is (d_1, d_2, d_3) given by (6),(7), (8) with θ as in (10).

Problem 10.

Solution. The proof has been discussed in class. Please refer to class notes. ■