Indian Institute of Science

E9-207: Basics of Signal Processing

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Solutions to Mid Term Exam #1, Spring 2018

Name and S.R. No.:

Instructions:

- Two sheets of paper are allowed.
- The time duration is 3 hrs.
- There are five main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and effort.
- Do not panic, do not cheat.
- Good luck!

Question No.	Points scored
1	
2	
3	
4	
5	
Total points	

PROBLEM 1: Examine if the following statements are true or false with correct reasoning. Random guessing or incorrect reasoning fetches zero credit. A statement is true if it is generic for all cases. A counter example is enough to make it false. All sub-parts of this problem carry equal credit.

1. Since downsampling and upsampling operations are not time invariant, all multi-rate systems that use downsamplers and upsamplers are non-LTI.

2. Let $\underline{v}_i = (a_{i1}, a_{i2}, \cdots, a_{iN})$ be a set of vectors for $i = 1, 2, \cdots N$. Let $\underline{u}_i = (a_{1i}, a_{2i}, \cdots, a_{Ni})$ be another set of vector $i = 1, 2, \cdots N$. We define two spaces $\mathcal{V}$ and $\mathcal{U}$ as $\mathcal{V} = \text{Span}(\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_N\})$ and $\mathcal{U} = \text{Span}(\{\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_N\})$. We know that $\dim(\mathcal{V}) < N$. Then $\dim(\mathcal{U}) > \dim(\mathcal{V})$.

3. The inverse of a stable filter is also stable.

4. Let a signal $s(t)$ be passed through a BIBO stable LTI system with impulse response $h(t)$ to get the output $y(t)$. Let E_s, E_h and E_y be the energies in $s(t)$, $h(t)$ and $y(t)$ respectively. Then $E_y \leq E_s E_h$.

5. Let $X(t)$ and $Y(t)$ be two independent W.S.S processes. Their linear combination $Z(t) = aX(t) + bY(t)$, $a, b \in \mathbb{R}$ is also a W.S.S. process.

(25 pts.)

SOLUTION:

Part 1: (False) Upsampling by rate 2 followed by rate 2 downsampling gives the original signal back. Therefore, this system is LTI.

Part 2: (False) $\mathcal{V}$ and $\mathcal{U}$ are row and column spaces of a matrix respectively, where a_{ij} are the elements of the matrix. We can disprove the claim with the example matrix $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$: $\underline{v}_1 = \underline{v}_2 = (1, 2)$. Therefore, $\dim(\mathcal{V}) = 1$. For this example, $\underline{u}_2 = 2\underline{u}_1 = (2, 2)$. Since $\underline{u}_1$ and $\underline{u}_2$ are linearly dependent, $\dim(\mathcal{U}) = 1$. Therefore, $\dim(\mathcal{U}) \not> \dim(\mathcal{V})$.

Part 3: (False) $H(z) = 2 - z^{-1}$ is FIR and hence stable. Its inverse $\frac{1}{2-z^{-1}}$ has a pole at $z = 0.5$ i.e., inside unit circle. Therefore the inverse is not sable.

Part 4: (False) There are several simple counter examples to disprove this:

Example 1: Let $s[n] = h[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases}$. This gives $y[n] = \begin{cases} 1, & n = 0, 2 \\ 2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$. Therefore, $E_s = E_h = 2$ and $E_y = 6 > E_s E_h = 4$.

Example 2: Let $s(t) = h(t) = \begin{cases} 1, & -1 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$. This gives $y(t) = \begin{cases} 2 - |t|, & -2 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$. Here, $E_s = E_h = 2$ and $E_y = \frac{16}{3} > E_s E_h = 4$.

Example 3: If $S(f)$, $H(f)$ and $Y(f)$ are the frequency responses of $s(t)$, $h(t)$ and $y(t)$ respectively, we have

$$Y(f) = S(f) H(f)$$

and

$$E_s = \int_{-\infty}^{\infty} |S(f)|^2 df, \quad E_h = \int_{-\infty}^{\infty} |H(f)|^2 df, \quad E_y = \int_{-\infty}^{\infty} |Y(f)|^2 df.$$

We prove using following example: $|S(f)|^2 = |H(f)|^2 = \begin{cases} f & 0 \leq f \leq 1, \\ 0 & \text{otherwise.} \end{cases}$ This gives us $E_s = E_h = \int_0^1 f df = \frac{1}{2}$ and $E_y = \int_0^1 |f|^2 df = \frac{1}{3}$. In this case $E_y = \frac{1}{3} \geq \frac{1}{4} = E_s E_h$.

Note 1: Using Cauchy-Schwartz inequality, we get

$$\left|\int_{-\infty}^{\infty} |Y(f)|\, df\right|^2 \le \left(\int_{-\infty}^{\infty} |S(f)|^2\, df\right)\left(\int_{-\infty}^{\infty} |H(f)|^2\, df = E_s E_h\right).$$

But, $E_y = \|Y(f)\|_{l_2}^2 = \int_{-\infty}^{\infty} |Y(f)|^2\, df$ which cannot be related to $\left|\int_{-\infty}^{\infty} |Y(f)|\, df\right|^2 = \|Y(f)\|_{l_1}^2$ as $\|Y(f)\|_{l_1}^2 \le \|Y(f)\|_{l_2}^2$ i.e., $\left|\int_{-\infty}^{\infty} |Y(f)|\, df\right|^2 \le E_y$. This does not give the desired result. From Cauchy-Schwartz inequality, the smallest value of $E_s E_h$ is $\|Y(f)\|_{l_1}^2$ which is achievable when $S(f) = kH(f)$, we can disprove the statement by choosing any arbitrary $S(f) = kH(f)$ where $|S(f)|$ is not a constant in f. Equivalently, we can take any time domain signals $s(t) = h(t)$ such that $|S(f)|$ is not a constant.

Part 5: (True) $\mathbb{E}[Z(t)] = a\mathbb{E}[X(t)] + b\mathbb{E}[X(t)] = a\mu_X + b\mu_Y$.

$$\begin{aligned}\mathbb{E}[Z(t) Z^*(t+\tau)] &= |a|^2 \mathbb{E}[X(t) X^*(t+\tau)] + ab^*\mathbb{E}[X(t) Y^*(t+\tau)] \\ &\quad + a^*b\mathbb{E}[Y(t) X^*(t+\tau)] + |b|^2 \mathbb{E}[Y(t) Y^*(t+\tau)] \\ &= |a|^2 R_X(\tau) + |b|^2 R_Y(\tau) + ab^*\mu_X\mu_Y^* + a^*b\mu_X^*\mu_Y.\end{aligned}$$

Since $\mathbb{E}[Z(t)]$ and $\mathbb{E}[Z(t) Z^*(t+\tau)]$ are independent of t, $Z(t)$ is also a W.S.S. process.

PROBLEM 2: This problem has two parts:

1. A discrete-time system with forcing function $f[n]$ and output $y[n]$ is represented using state variables $u[n]$ and $w[n]$ as

$$\begin{aligned} w[n+1] &= 2u[n] + 3f[n], \\ u[n+1] &= w[n] + 2f[n], \\ y[n] &= u[n] + 3w[n] + f[n]. \end{aligned}$$

What are the modes of the system? What is the the transfer function of the system? What are the state space parameters ($\mathbf{A}$, $\underline{b}$, $\underline{c}$, d) of the system? (10 pts.)

2. Consider a cascade of two LTI systems A and B with impulse responses $H_A(z) = \frac{1-z^{-1}}{(2+z^{-1})(1-3z^{-1})}$ and $H_B(z) = \frac{1-3z^{-1}}{1-z^{-1}}$ respectively. Write down the time difference equations representing the systems A and B. Combine the two difference equations to obtain a time difference equation for the overall cascaded system. Compare the obtained equation with the overall impulse response of the cascaded system. (10 pts.)

SOLUTION:

Part 1: Taking z-transform of the given equations, we get

$$zW(z) = 2U(z) + 3F(z) \tag{1}$$

$$zU(z) = W(z) + 2F(z) \tag{2}$$

$$Y(z) = U(z) + 3W(z) + F(z). \tag{3}$$

Multiplying (2) by z on both sides and substituting $zW(z)$ from (1), we get

$$z^2U(z) = zW(z) + 2zF(z) = 2U(z) + (2z+3)F(z)$$

$$\implies U(z) = \frac{2z+3}{z^2-2}F(z) \tag{4}$$

Using this in (2) to obtain $W(z)$:

$$W(z) = zU(z) - 2F(z) = \frac{z(2z+3) - 2(z^2-2)}{z^2-2}F(z) = \frac{3z+4}{z^2-2}F(z). \tag{5}$$

Using (4) and (5) in (3), we get

$$\frac{Y(z)}{F(z)} = \frac{(2z+3) + 3(3z+4) + z^2 - 2}{z^2-2} = \frac{z^2+11z+13}{z^2-2} = \frac{1+11z^{-1}+13z^{-2}}{1-2z^{-2}}.$$

Therefore, the transfer function of the system is $\frac{1+11z^{-1}+13z^{-2}}{1-2z^{-2}}$. The modes of the system are $\pm\sqrt{2}$.

Defining $\underline{v}[n] = \begin{bmatrix} w[n] \\ u[n] \end{bmatrix}$, we can rewrite the given set of equations as

$$\underline{v}[n+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}}_{\mathbf{A}} \underline{v}[n] + \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\underline{b}} f[n], \text{ and } y[n] = \underbrace{\begin{bmatrix} 3 & 1 \end{bmatrix}}_{\underline{c}^T} \underline{v}[n] + \underbrace{1}_{d} f[n].$$

Therefore, the state variables of the system are $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\underline{c}^T = \begin{bmatrix} 3 & 1 \end{bmatrix}$, $d = 1$.

Part 2: Let $x[n]$ be the input to the cascaded system, $w[n]$ be the output of the system A, and $y[n]$ be the output of the entire cascaded system. From the impulse response of the systems, we have

$$\frac{W(z)}{X(z)} = \frac{1 - z^{-1}}{(2 + z^{-1})(1 - 3z^{-1})}, \qquad \frac{Y(z)}{W(z)} = \frac{1 - 3z^{-1}}{1 - z^{-1}}.$$
$$\implies \left(2 - 5z^{-1} - 3z^{-2}\right) W(z) = \left(1 - z^{-1}\right) X(z)$$
$$\text{and } \left(1 - z^{-1}\right) Y(z) = \left(1 - 3z^{-1}\right) W(z)$$

Taking the inverse z-transform of the above equations, we get the time difference equations as

$$2w[n] - 5w[n-1] - 3w[n-2] = x[n] - x[n-1] \tag{6}$$
$$y[n] - y[n-1] = w[n] - 3w[n-1] \tag{7}$$

Similarly, the transfer function of the overall cascaded system is $H_A(z) H_B(z) = \frac{1}{2+z^{-1}}$. Therefore, the time-difference equation of the cascaded system is

$$2y[n] + y[n-1] = x[n]$$
$$\implies 2y[n] + y[n-1] - x[n] = 0 \tag{8}$$

Equation (7) is true for all n. Therefore,

$$w[n] - 3w[n-1] = y[n] - y[n-1], \tag{9}$$
$$\text{and } w[n-1] - 3w[n-2] = y[n-1] - y[n-2]. \tag{10}$$

Equation (6) can be written as

$$2(w[n] - 3w[n-1]) + (w[n-1] - 3w[n-2]) = x[n] - x[n-1].$$

Using (9) and (10) in the above equation, we get

$$2(y[n] - y[n-1]) + y[n-1] - y[n-2] = x[n] - x[n-1]$$
$$\implies 2y[n] - y[n-1] - y[n-2] = x[n] - x[n-1] \tag{11}$$

Equation (11) gives the time difference equation of the cascaded system. The equation (11) can be written as

$$(2y[n] + y[n-1] - x[n]) - (2y[n-1] + y[n-2] - x[n-1]) = 0 \tag{12}$$

We can see similarity in (8) and (12). If we define $u[n] = 2y[n] + y[n-1] - x[n]$, the equation (12) becomes $u[n] = u[n-1]$. Therefore $u[n] = c$ for some constant c. In order to prove that (8) and (12) are same, we have to show that $c = 0$ is the only possibility.

Since the ROC of $\sum_{k=0}^{\infty} z^{-k}$ is $|z| < 1$ and the ROC of $\sum_{k=-\infty}^{1} z^{-k}$ is $|z| > 1$, we claim that $c = 0$ is the only value for which $W(z) = \sum_{k=-\infty}^{\infty} cz^{-k}$ can be defined on z plane. Therefore, $c = 0 \implies u[n] = 0$. Hence, we can say that (12) implies (8).

Problem 3: This problem has two parts:

1. Let $\mathcal{S}$ be a finite dimensional vector space and $\mathcal{V}_1 \subset \mathcal{S}$ and $\mathcal{V}_2 \subset \mathcal{S}$ be two sub-spaces in $\mathcal{S}$. Show that $\dim(\mathcal{V}_1 + \mathcal{V}_2) = \dim(\mathcal{V}_1) + \dim(\mathcal{V}_2) - \dim(\mathcal{V}_1 \cap \mathcal{V}_2)$. (13 pts.)

2. Consider the following three signals given by

$$s_1(t) = \begin{cases} 1, & 0 \le t \le 0.25, \\ -1. & 0.25 < t \le 0.75, \\ 1, & 0.75 < t \le 1, \\ 0, & \text{otherwise}, \end{cases} \quad s_2(t) = \begin{cases} 1, & 0 \le t \le 0.5, \\ -1, & 0.5 < t \le 1 \\ 0 & \text{otherwise}, \end{cases}, \quad s_3(t) = \begin{cases} t - 0.5, & 0 \le t \le 1, \\ 0 & \text{otherwise}, \end{cases}.$$

Find an appropriate orthonormal basis for the signal space spanned by the three signals and represent the three signals as points in the signal space. What is the least squares approximation of the signal

$$s(t) = \begin{cases} \sin(2\pi t) & 0 \le t \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

in the signal space. Plot the approximated signal as a function of time. (12 pts.)

Solution:

Part 1:

We know that $\mathcal{V}_1 \cap \mathcal{V}_2$ is a sub-space. Let $R = \{\underline{y}_1, ..., \underline{y}_l\}$ be an orthogonal basis of $\mathcal{V}_1 \cap \mathcal{V}_2$. Every finite linearly independent subset of $\mathcal{V}$ can be extended to a basis of $\mathcal{V}$. Let $S = \{\underline{u}_1, ..., \underline{u}_m\}$ be an orthogonal set such that $S \cup R$ is an orthogonal basis for $\mathcal{V}_1$. Similarly, let $T = \{\underline{w}_1, ..., \underline{w}_n\}$ be an orthogonal set such that $T \cup R$ is an orthogonal basis for $\mathcal{V}_2$. Therefore, $\dim(\mathcal{V}_1) = l + m$ and $\dim(\mathcal{V}_2) = l + n$. Our goal is to show that

$$S \cup T \cup R = \{\underline{y}_1, ..., \underline{y}_l, \underline{u}_1, ..., \underline{u}_m, \underline{w}_1, ..., \underline{w}_n\} \text{ is a basis of } \mathcal{V}_1 + \mathcal{V}_2. \tag{13}$$

We need to show that $\mathcal{V}_1 + \mathcal{V}_2 = \text{span}(S \cup T \cup R)$ and $S \cup T \cup R$ is a linearly independent set. Assuming that this has been shown, then

$$\dim(\mathcal{V}_1 + \mathcal{V}_2) + \dim(\mathcal{V}_1 \cap \mathcal{V}_2) = (l + m + n) + l = (l + m) + (l + n) = \dim(\mathcal{V}_1) + \dim(\mathcal{V}_2).$$

Let us now show the result in equation (13). Let $\underline{v} \in \mathcal{V}_1 + \mathcal{V}_2$. Then $\underline{v} = \underline{v}_1 + \underline{v}_2$ where $\underline{v}_1 \in \mathcal{V}_1$, and $\underline{v}_2 \in \mathcal{V}_2$. Since $\underline{v}_1 \in \text{span}(S \cup R)$ and $\underline{v}_2 \in \text{span}(T \cup R)$, it follows that $\underline{v} \in \text{span}(S \cup T \cup R)$ and so $\mathcal{V}_1 + \mathcal{V}_2 = \text{span}(S \cup T \cup R)$. Suppose that

$$\sum_{i=1}^{l} a_i \underline{y}_i + \sum_{i=1}^{m} b_i \underline{u}_i + \sum_{i=1}^{n} \underline{c}_i w_i = 0$$

where $a_i, b_i, c_i \in \mathbb{R}$. Then,

$$\underbrace{\sum_{i=1}^{n} c_i \underline{w}_i}_{\in \mathcal{V}_2} = \underbrace{-\sum_{i=1}^{l} a_i \underline{y}_i - \sum_{i=1}^{m} b_i \underline{u}_i}_{\in \mathcal{V}_!} \in \mathcal{V}_1 \cap \mathcal{V}_2.$$

Thus, $\sum_{i=1}^{n} c_i \underline{w}_i = \sum_{i=1}^{l} d_i \underline{y}_i$, since R is a basis of $\mathcal{V}_1 \cap \mathcal{V}_2$. Since $T \cup R$ is a basis of $\mathcal{V}_2$, it follows that $c_i = 0$, $1 \le i \le n$, and $d_i = 0$, $1 \le i \le l$. Now we have $\sum_{i=1}^{l} a_i \underline{y}_i + \sum_{i=1}^{m} b_i \underline{u}_i = 0$. Since $S \cup R$ is a basis of $\mathcal{V}_1$, it follows that $a_i = 0,\ 1 \leq i \leq l$ and $b_i = 0,\ 1 \leq i \leq m$. This shows $S \cup T \cup R$ is a linearly independent set and so $S \cup T \cup R$ is a basis of $\mathcal{V}_1 + \mathcal{V}_2$.

Part 2:

We proceed with Gram-Schmidt orthonormalization of the signals to find an orthonormal basis.

$$v_1(t) = s_1(t)$$

$$v_2(t) = s_2(t) - \frac{\langle s_2(t), v_1(t)\rangle}{\langle v_1(t), v_1(t)\rangle} v_1(t)$$

Now,

$$\begin{aligned}
\langle s_2(t), v_1(t)\rangle &= \int_0^1 s_2(t) v_1(t) dt \\
&= \int_0^{0.25} (1 \times 1) dt + \int_{0.25}^{0.5} (-1 \times 1) dt + \int_{0.5}^{0.75} (-1 \times -1) dt + \int_{0.75}^{1} (1 \times -1) dt \\
&= 0.25 - 0.25 + 0.25 - 0.25 \\
&= 0
\end{aligned}$$

So we get,

$$v_2(t) = s_2(t)$$

Now,

$$v_3(t) = s_3(t) - \frac{\langle s_3(t), v_1(t)\rangle}{\langle v_1(t), v_1(t)\rangle} v_1(t) - \frac{\langle s_3(t), v_2(t)\rangle}{\langle v_2(t), v_2(t)\rangle} v_2(t)$$

Let us find $\langle s_3(t), v_1(t)\rangle$ and $\langle s_3(t), v_2(t)\rangle$

$$\begin{aligned}
\langle s_3(t), v_1(t)\rangle &= \int_0^1 s_3(t) v_1(t) dt \\
&= \int_0^{0.25} ((t-0.5) \times 1) dt + \int_{0.25}^{0.75} ((t-0.5) \times -1) dt + \int_{0.75}^{1} ((t-0.5) \times 1) dt \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle s_3(t), v_2(t)\rangle &= \int_0^1 s_3(t) v_2(t) dt \\
&= \int_0^{0.5} ((t-0.5) \times 1) dt + \int_{0.5}^{1} ((t-0.5) \times -1) dt \\
&= -0.25
\end{aligned}$$

$$\begin{aligned}
\langle v_2(t), v_2(t)\rangle &= \int_0^{0.5} (1 \times 1) dt + \int_{0.5}^{1} (-1 \times -1) dt \\
&= 0.5 + 1 - 0.5 \\
&= 1
\end{aligned}$$

So, we get

$$v_3(t) = s_3(t) + 0.25v_2(t) = s_3(t) + 0.25s_2(t)$$

$$v_3(t) = \begin{cases} t - 0.25, & 0 \le t \le 0.5, \\ t - 0.75, & 0.5 < t \le 1 \\ 0 & \text{otherwise,} \end{cases},$$

$v_1(t)$, $v_2(t)$ and $v_3(t)$ are orthogonal basis for the signal space spanned by $s_1(t)$, $s_2(t)$ and $s_3(t)$. We need to normalise the vectors to get and orthonormal basis.

We get orthonormal basis $u_1(t)$, $u_2(t)$ and $u_3(t)$ as follows

$$u_1(t) = \frac{v_1(t)}{||v_1(t)||} = v_1(t) = s_1(t)$$

$$u_2(t) = \frac{v_2(t)}{||v_2(t)||} = v_2(t) = s_2(t)$$

$$\begin{aligned} u_3(t) &= \frac{v_3(t)}{||v_3(t)||} \\ &= \frac{v_3(t)}{\sqrt{\int_0^1 (v_3(t))^2 dt}} \\ &= \frac{v_3(t)}{\sqrt{\int_0^{0.5} (t-0.25)^2 dt + \int_{0.5}^1 (t-0.75)^2 dt}} \\ &= \frac{v_3(t)}{\sqrt{\frac{1}{48}}} \\ &= \sqrt{48}(s_3(t) + 0.25s_2(t)) \\ &= \sqrt{3}(4s_3(t) + s_2(t)) \end{aligned}$$

From above we can write the following,

$$s_1(t) = u_1(t),\ s_2(t) = u_2(t)\ and\ s_3(t) = -0.25u_2(t) + \frac{1}{\sqrt{48}}u_3(t)$$

The signals $s_1(t)$, $s_2(t)$ and $s_3(t)$ can be represented as points in signal space with respect orthonormal basis $\{u_1(t), u_2(t), u_3(t)\}$ as

$$s_1(t) = (1, 0, 0),\ s_2(t) = (0, 1, 0)\ and\ s_3(t) = (0, -0.25, \frac{1}{\sqrt{48}})$$

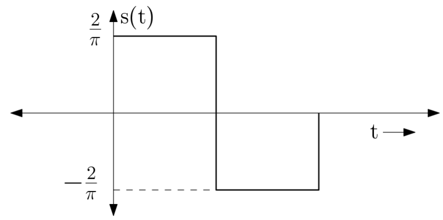
To find the least squared approximation of signal $s(t)$ in the signal space,

$$\begin{aligned} s(t) &= \langle s(t), u_1(t)\rangle u_1(t) + \langle s(t), u_2(t)\rangle u_2(t) + \langle s(t), u_3(t)\rangle u_3(t) \\ &= (\int_0^1 s(t)u_1(t)dt)u_1(t) + (\int_0^1 s(t)u_2(t)dt)u_2(t) + (\int_0^1 s(t)u_3(t)dt)u_3(t) \\ &= 0 \times u_1(t) + \frac{2}{\pi} \times u_2(t) + 0 \times u_3(t) \\ &= \frac{2}{\pi}u_2(t) \end{aligned}$$

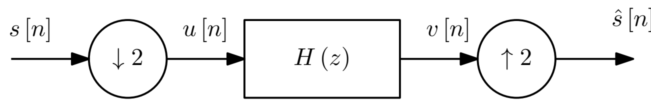
Therefore the least squared approximation of the signal,

$$s(t) = \begin{cases} \frac{2}{\pi}, & 0 \leq t \leq 0.5, \\ -\frac{2}{\pi}, & 0.5 < t \leq 1 \\ 0 & \text{otherwise,} \end{cases},$$

The plot of the least squared approximation of signal $s(t)$ is shown below,



Problem 4: A signal $s(t)$ with 60 Hz bandwidth and sampled at 600 Hz to obtain the samples $s[n]$. The signal $s[n]$ is passed through following operations where $H(z)$ is a low-pass filter. How do you choose the passband and stop band frequencies of $H(z)$ such that the filter order is minimized and $\hat{s}[n] = s[n]$? Sketch the frequency responses of the signal at various stages in the system.

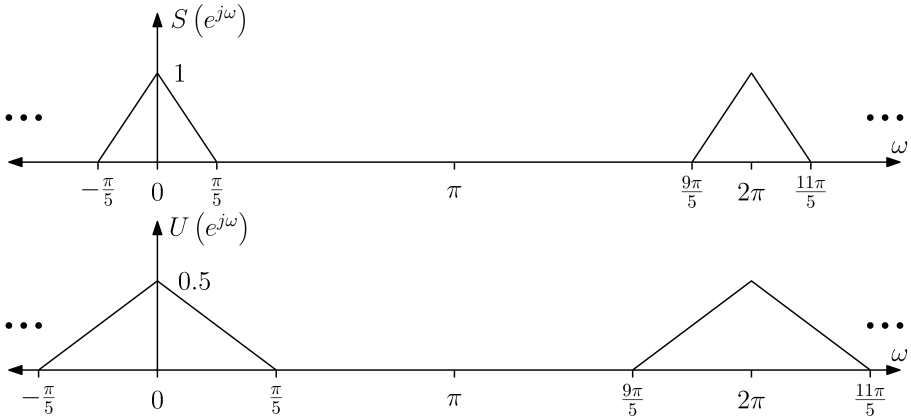


(15 pts.)

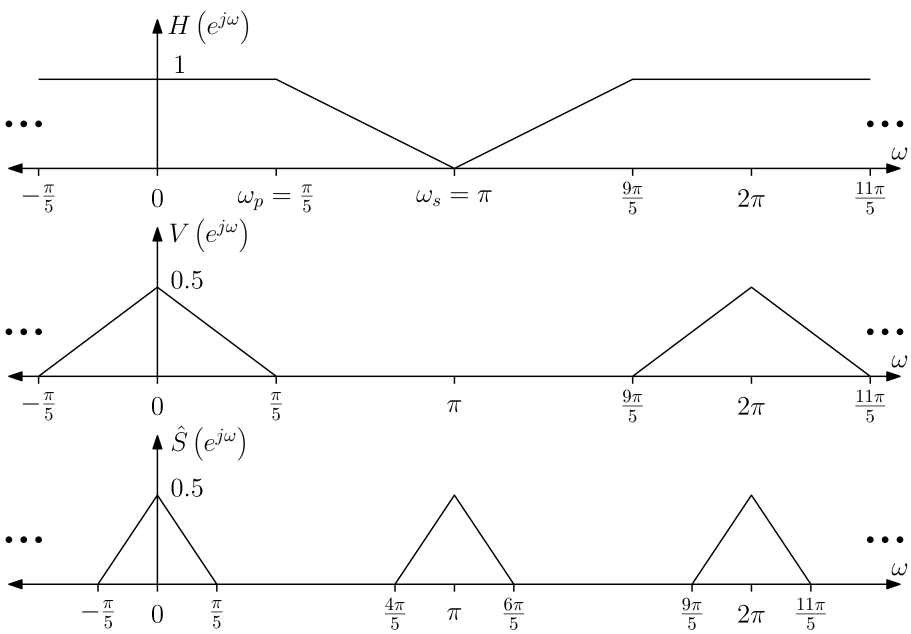
Solution:

Since $\hat{s}[n]$ is an output of upsampler, $\hat{s}[n] = 0$ for odd n. Therefore, the given circuit cannot reconstruct the original signal.

The bandwidth of the signal is $\omega_B = 2\pi\frac{60}{600} = \frac{\pi}{5}$. The frequency responses of $s[n]$ and $u[n]$ are given below



In order to minimize the order of $H(z)$ its transition bandwidth must be as large as possible. Therefore, the passband frequency is $\omega_s = \frac{\pi}{2}$. The stopband frequency is π. Following gives the frequency response of $H(z)$ and the signal $v[n]$ and $\hat{s}[n]$.



PROBLEM 5: Consider the following fractional sampling rate converter that increases the sampling rate by a factor 1.5. Efficient architectures of the same circuit can be obtained in two ways:

1. Architecture-1: Represent $H(z)$ using type-2 polyphase components $\left\{E_0\left(z^3\right), E_1\left(z^3\right), E_2\left(z^3\right)\right\}$ to obtain an efficient architecture for the interpolation stage. Next, we represent each $E_i(z)$ using type-1 polyphase components $\left\{E_{i0}\left(z^2\right), E_{i1}\left(z^2\right)\right\}$ to obtain an efficient architecture for the decimation filters.

2. Architecture-2: Represent $H(z)$ using type-1 polyphase components $\left\{R_0\left(z^2\right), R_1\left(z^2\right)\right\}$ to obtain an efficient architecture for the decimation stage. Next, we represent each $R_i(z)$ using polyphase components $\left\{R_{i0}\left(z^3\right), R_{i1}\left(z^3\right), R_{i2}\left(z^3\right)\right\}$ to obtain an efficient architecture for the interpolation filters.

Express $H(z)$ using the filters $E_{ij}(z)$ from Architecture-1. Similarly, express $H(z)$ using the filters $R_{ij}(z)$ from Architecture-2. How are the filters $E_{ij}(z)$ and $R_{ij}(z)$ from the two architectures related? (15 pts.)

SOLUTION:

Architecture-1: Using type-2 polyphase decomposition of $H(z)$ for rate 3, we have

$$H(z) = z^{-2}E_0\left(z^3\right) + z^{-1}E_1\left(z^3\right) + E_2\left(z^3\right). \tag{14}$$

Using rate-2, type-1 decomposition on $E_i(z)$, we have

$$E_i(z) = E_{i0}\left(z^2\right) + z^{-1}E_{i1}\left(z^2\right), \quad i = 0, 1, 2. \tag{15}$$

Using (15) in (14), we get

$$H(z) = z^{-2}\left[E_{00}\left(z^6\right) + z^{-3}E_{01}\left(z^6\right)\right] + z^{-1}\left[E_{10}\left(z^6\right) + z^{-3}E_{11}\left(z^6\right)\right] + E_{20}\left(z^6\right) + z^{-3}E_{21}\left(z^6\right)$$
$$H(z) = E_{20}\left(z^6\right) + z^{-1}E_{10}\left(z^6\right) + z^{-2}E_{00}\left(z^6\right) + z^{-3}E_{21}\left(z^6\right) + z^{-4}E_{11}\left(z^6\right) + z^{-5}E_{01}\left(z^6\right). \tag{16}$$

Architecture-2: Using type-1 polyphase decomposition of $H(z)$ for rate 2, we have

$$H(z) = R_0\left(z^2\right) + z^{-1}R_1\left(z^2\right). \tag{17}$$

Using rate-3, type-2 decomposition on $R_i(z)$, we have

$$R_i(z) = z^{-2}R_{i0}\left(z^3\right) + z^{-1}R_{i1}\left(z^3\right) + R_{i2}\left(z^3\right), \quad i = 0, 1, 2. \tag{18}$$

Using (18) in (17), we get

$$H(z) = z^{-4}R_{00}\left(z^6\right) + z^{-2}R_{01}\left(z^6\right) + R_{02}\left(z^6\right) + z^{-1}\left[z^{-4}R_{10}\left(z^6\right) + z^{-2}R_{11}\left(z^6\right) + R_{12}\left(z^6\right)\right]$$
$$H(z) = R_{02}\left(z^6\right) + z^{-1}R_{12}\left(z^6\right) + z^{-2}R_{01}\left(z^6\right) + z^{-3}R_{11}\left(z^6\right) + z^{-4}R_{00}\left(z^6\right) + z^{-5}R_{10}\left(z^6\right). \tag{19}$$

Comparing the coefficients of z^{6n}, $n \in \mathbb{Z}$ in (16) and (19), we get $E_{20}\left(z^6\right) = R_{02}\left(z^6\right)$ i.e, $E_{20}(z) = R_{02}(z)$. Similarly, comparing the coefficients of z^{6n+1}, $n \in \mathbb{Z}$ in (16) and (19), we get $E_{10}(z) = R_{12}(z)$. Continuing further comparing the coefficients of z^{6n+k}, $n \in \mathbb{Z}$ for $k = 0, 1, 2, 3, 4, 5$, we get

$$\begin{aligned}
k = 0 &\implies E_{20}(z) = R_{02}(z), \\
k = 1 &\implies E_{10}(z) = R_{12}(z), \\
k = 2 &\implies E_{00}(z) = R_{01}(z), \\
k = 3 &\implies E_{21}(z) = R_{11}(z), \\
k = 4 &\implies E_{11}(z) = R_{00}(z), \\
k = 5 &\implies E_{01}(z) = R_{10}(z).
\end{aligned}$$