Indian Institute of Science
E9−207: Basics of Signal Processing
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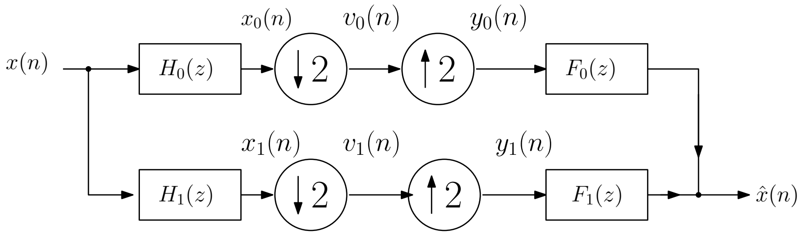
Homework #4 Solutions, Spring 2018
Solutions prepared by Arijit and Ankur

PROBLEM 1:
Consider the two channel filter bank.
(a) Obtain the conditions on the synthesis filter banks to force the aliasing to zero. (1 point)
(b) Let $H_0(z) = 1 + z^{-1}$ and $H_1(z) = 1 - z^{-1}$. Construct synthesis filter banks which ensure perfect reconstruction of the input signal. (1 point)
Solution:

We note that



$$
\begin{aligned}
Y_0(z) &= V_0(z^2) = \frac{1}{2}\big(X_0(z) + X_0(-z)\big)\\
Y_1(z) &= V_1(z^2) = \frac{1}{2}\big(X_1(z) + X_1(-z)\big)\\
\hat{X}(z) &= Y_0(z)F_0(z) + Y_1(z)F_1(z)\\
&= \frac{1}{2}X(z)\big(H_0(z)F_0(z) + H_1(z)F_1(z)\big) + \frac{1}{2}X(-z)\big(H_0(-z)F_0(z) + H_1(-z)F_1(z)\big)
\end{aligned}
$$

(a) To force the alias term to zero, we let

$$F_0(z) = R(z)H_1(-z); F_1(z) = -R(z)H_0(-z) \text{ for some causal stable } R(z).$$

(b) For perfect reconstruction, we let

$$F_0(z) = 1 + z^{-1}; F_1(z) = -1 + z^{-1}$$

which gives us $\hat{x}(n) = 4x(n-1)$.

PROBLEM 2:
Obtain the Haar wavelet decomposition for the signal $f(t)$ using the Haar basis. Indicate the signal dimension at each subspace. Sketch the waveforms explicitly at each subspace. Obtain the reconstructed signal in functional form after nulling out any spike of (1/8)th unit of time. Analyze using Fourier Transform. How much of energy is lost in the recovered signal?(8 points)

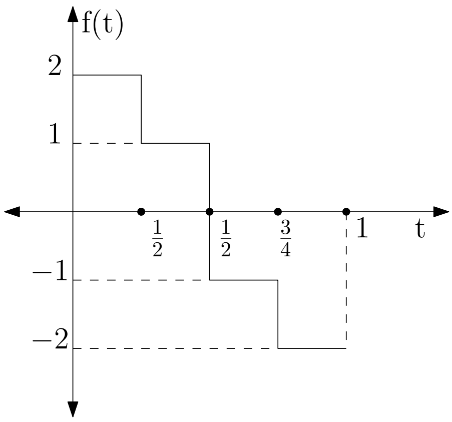
$$f(t) = \begin{cases} 2 & 0 \le t < \frac{1}{4} \\ 1 & \frac{1}{4} \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < \frac{3}{4} \\ -2 & \frac{3}{4} \le t < 1 \end{cases}$$

Solution:

Given

$$f(t) = \begin{cases} 2 & 0 \le t < \frac{1}{4} \\ 1 & \frac{1}{4} \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < \frac{3}{4} \\ -2 & \frac{3}{4} \le t < 1 \end{cases}$$

It can be represented in graphical form as,



f(t) can be written as,

$$f(t) = 2\phi(4t) + \phi(4t-1) - \phi(4t-2) - 2\phi(4t-3) \tag{1}$$

We will use the identities,

$$\phi(2^j t - 2k) = \frac{1}{2}\{\phi(2^{j-1}t - k) + \psi(2^{j-1}t - k)\} \tag{2}$$

$$\phi(2^j t - (2k+1)) = \frac{1}{2}\{\phi(2^{j-1}t - k) - \psi(2^{j-1}t - k)\} \tag{3}$$

Using the above identities in eq. 1 we get,

$$f(t) = [\phi(2t) + \psi(2t)\}] + \frac{1}{2}[\phi(2t) - \psi(2t)] - \frac{1}{2}[\phi(2t-1) + \psi(2t-1)] - [\phi(2t-1) - \psi(2t-1)] \tag{4}$$

$$= \frac{3}{2}\phi(2t) + \frac{1}{2}\psi(2t) - \frac{3}{2}\phi(2t-1) + \frac{1}{2}\psi(2t-1) \tag{5}$$

Using identities 2 and 3 in eq. 5 we get,

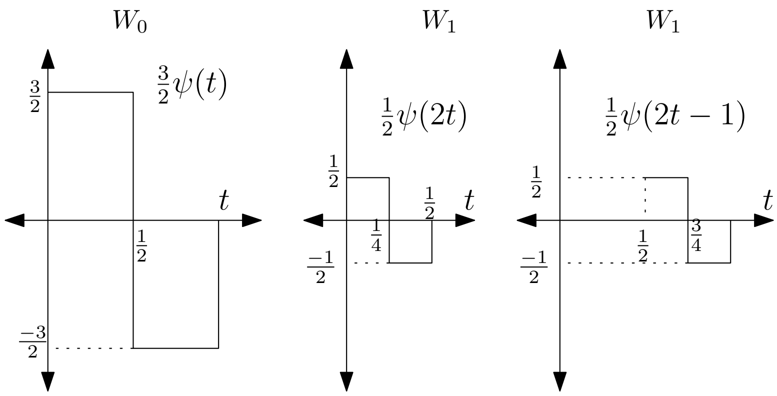
$$f(t) = \frac{3}{4}[\phi(t) + \psi(t)] + \frac{1}{2}\psi(2t) - \frac{3}{4}[\phi(t) - \psi(t)] + \frac{1}{2}\psi(2t-1) \tag{6}$$

$$= \frac{3}{2}\psi(t) + \frac{1}{2}\psi(2t) + \frac{1}{2}\psi(2t-1) \tag{7}$$

We can see that $\frac{3}{2}\psi(t)$ belongs to W_0. $\frac{1}{2}\psi(2t)$ and $\frac{1}{2}\psi(2t-1)$ belongs to W_1.

Therefore dim(V_0)=0, dim(W_0)=1 and dim(W_1)=2. Dimensions of all higher subspaces are 0.

The wavefprms at each subspace is shown below,



Any spike of duration $\frac{1}{8}$ will belong to W_2. But we can see that dim(W_2)=0. So the reconstructed signal will be the original signal itself. We can verify it using the following identities which are obtained from eq. 2 and 3,

$$\phi(2^j t - k) = \phi(2^{j+1}t - 2k) + \phi(2^{j+1}t - (2k+1)) \tag{8}$$

$$\psi(2^j t - k) = \phi(2^{j+1}t - 2k) - \phi(2^{j+1}t - (2k+1)) \tag{9}$$

Using above in eq. 7, we can verify that reconstructed signal $g(t)$ is

$$g(t) = 2\phi(4t) + \phi(4t-1) - \phi(4t-2) - 2\phi(4t-3)$$

As the reconstructed signal is the original signal itself, energy lost = 0.

Fourier analysis:

$$g(t) = 2\phi(4t) + \phi(4t-1) - \phi(4t-2) - 2\phi(4t-3)$$

$$\implies G(f) = \frac{1}{2}\text{sinc}\left(\frac{\text{f}}{4}\right)e^{\frac{-j2\pi f}{8}} + \frac{1}{4}\text{sinc}\left(\frac{f}{4}\right)e^{\frac{-j6\pi f}{8}} - \frac{1}{4}\text{sinc}\left(\frac{f}{4}\right)e^{\frac{-j10\pi f}{8}} - \frac{1}{2}\text{sinc}\left(\frac{f}{4}\right)e^{\frac{-j14\pi f}{8}}$$

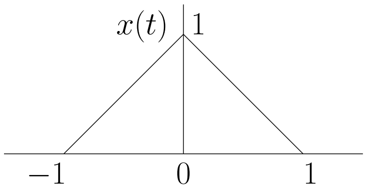
PROBLEM 3:
Consider the signal

$$x(t) = \begin{cases} 1 - |t| & \text{for } -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Obtain the projection of $x(t)$ on V_0 and W_0 spaces of Haar multi resolution analysis. Is the projection shift invariant? (4 points)

(b) Compute $\sum_{n=-\infty}^{\infty} \phi(t-n)$. (1 point)

Solution:



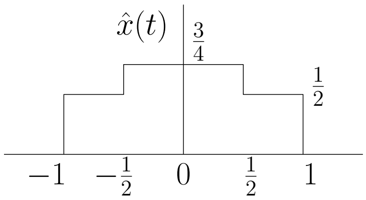
(a) V_0 space is spanned by the interger shifts of $\phi(t)$ and W_0 space is spanned by the integer shifts of $\psi(t)$. Projecting $x(t)$ on V_0 requires $\phi(t)$ and $\phi(t+1)$ while that on W_0 requires $\psi(t)$ and $\psi(t+1)$. Therefore, let the projection be

$$\hat{x}(t) = a_0\phi(t) + a_{-1}\phi(t+1) + b_0\psi(t) + b_{-1}\psi(t+1)$$

The coefficients can be determined by using projections on the basis functions:

$$\begin{aligned} a_0 &= \int_0^1 x(t)\phi(t) = \frac{1}{2} \\ a_{-1} &= \int_{-1}^0 x(t)\phi(t+1) = \frac{1}{2} \\ b_0 &= \int_0^1 x(t)\psi(t) = \frac{1}{4} \\ b_{-1} &= \int_0^1 x(t)\psi(t+1) = -\frac{1}{4} \\ \therefore \hat{x}(t) &= \frac{1}{2}\phi(t) + \frac{1}{2}\phi(t+1) + \frac{1}{4}\psi(t) - \frac{1}{4}\psi(t+1) \end{aligned}$$

The projected signal looks as follows.



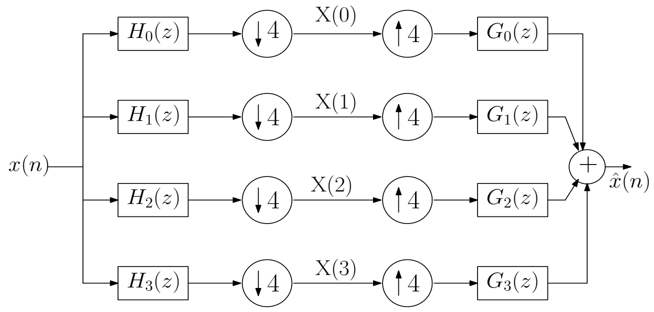
Projection is shift invariant for integer shifts.

(b)Summing the integer shifted versions of the scaling functions, we obtain $\sum_{n=-\infty}^{\infty} \phi(t-n) = 1$.

PROBLEM 4:
The normalized DFT of an N length sequence is defined as follows:

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

We wish to compute the normalized DFT $\{X(0), X(1), X(2), X(3)\}$ of a length 4 sequence using the 4 channel filter bank shown below:

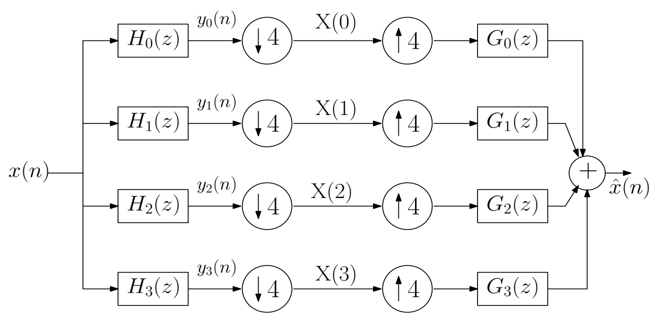


(a) Find the analysis filters $\{h_i(n)\}_{i=0}^{3}$ and synthesis filters $\{g_i(n)\}_{i=0}^{3}$ used to implement this filter bank. (8 points) (b) If the analysis filters are to be made causal, what is the delay introduced by the system? (2 points)

Solution:
(a) Let $N = 4$.

$$X(k) = \frac{1}{\sqrt{4}} \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}kn} = \frac{1}{2} \sum_{n=0}^{3} x(n) e^{-j\frac{\pi}{2}kn}$$

$$\begin{aligned}
X(0) &= \frac{1}{2}x(0) + \frac{1}{2}x(1) + \frac{1}{2}x(2) + \frac{1}{2}x(3) \\
X(1) &= \frac{1}{2}x(0) - j\frac{1}{2}x(1) - \frac{1}{2}x(2) + j\frac{1}{2}x(3) \\
X(2) &= \frac{1}{2}x(0) - \frac{1}{2}x(1) + \frac{1}{2}x(2) - \frac{1}{2}x(3) \\
X(3) &= \frac{1}{2}x(0) + j\frac{1}{2}x(1) - \frac{1}{2}x(2) - j\frac{1}{2}x(3)
\end{aligned} \tag{10}$$



Look at $X(k)$ and $y_k(n), k = 0, 1, 2, 3$.

$$y_k(n) = \sum_{m=0}^{3} x(m)h_k(n-m), k = 0, 1, 2, 3$$

After downsampling, only sample corresponding to $n = 0$ will go through futher system.We have

$$\begin{aligned}
X(0) = y_0(n) = x(0)h_0(0) + x(1)h_0(-1) + x(0)h_0(-2) + x(0)h_0(-3) \\
X(1) = y_1(n) = x(0)h_1(0) + x(1)h_1(-1) + x(0)h_1(-2) + x(0)h_1(-3) \\
X(2) = y_3(n) = x(0)h_2(0) + x(1)h_2(-1) + x(0)h_2(-2) + x(0)h_2(-3) \\
X(3) = y_3(n) = x(0)h_3(0) + x(1)h_3(-1) + x(0)h_3(-2) + x(0)h_3(-3)
\end{aligned} \tag{11}$$

Comparing equations 10 and 11, we get

$$h_0(n) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}} \end{bmatrix}$$

$$h_1(n) = \begin{bmatrix} \frac{j}{2} & -\frac{1}{2} & -\frac{j}{2} & \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}} \end{bmatrix}$$

$$h_2(n) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}} \end{bmatrix}$$

$$h_3(n) = \begin{bmatrix} -\frac{j}{2} & -\frac{1}{2} & \frac{j}{2} & \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}} \end{bmatrix}$$

After upsampling, we have the following signals going through the synthesis filters:

$$\begin{bmatrix} X(0) & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} X(1) & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} X(2) & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} X(3) & 0 & 0 & 0 \end{bmatrix}$$

The output

$$\hat{x}(n) = X(0)g_0(n) + X(1)g_1(n) + X(2)g_2(n) + X(3)g_3(n) \tag{12}$$

For perfect reconstruction, $\hat{x}(n) = x(n)$. The normalized IDFT is given by

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}kn} = \frac{1}{\sqrt{4}} \sum_{k=0}^{3} X(k)e^{j\frac{2\pi}{4}kn} = \frac{1}{2} \sum_{k=0}^{3} X(k)e^{j\frac{\pi}{2}kn} \tag{13}$$

$$
\begin{aligned}
x(0) &= \frac{1}{2}X(0) + \frac{1}{2}X(1) + \frac{1}{2}X(2) + \frac{1}{2}X(3) \\
x(1) &= \frac{1}{2}X(0) + j\frac{1}{2}X(1) - \frac{1}{2}X(2) - j\frac{1}{2}X(3) \\
x(2) &= \frac{1}{2}X(0) - \frac{1}{2}X(1) + \frac{1}{2}X(2) - \frac{1}{2}X(3) \\
x(3) &= \frac{1}{2}X(0) - j\frac{1}{2}X(1) - \frac{1}{2}X(2) + j\frac{1}{2}X(3)
\end{aligned} \tag{14}
$$

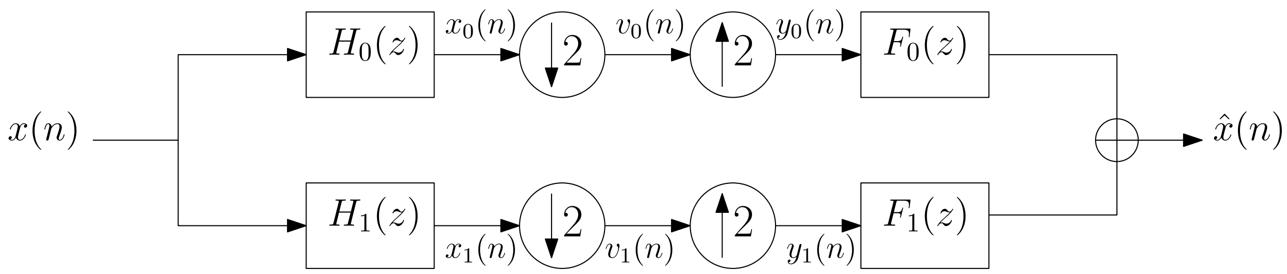
Comparing equations 13 and 14, we get

$$
\begin{aligned}
g_0(n) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}} \end{bmatrix} \\
g_1(n) &= \begin{bmatrix} -\frac{j}{2} & -\frac{1}{2} & \frac{j}{2} & \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}} \end{bmatrix} \\
g_2(n) &= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}} \end{bmatrix} \\
g_3(n) &= \begin{bmatrix} \frac{j}{2} & -\frac{1}{2} & -\frac{j}{2} & \underset{\substack{\uparrow \\ n=0}}{\frac{1}{2}} \end{bmatrix}
\end{aligned}
$$

(b) From equation 12, we note that delay is due to the $\{g_i(n)\}_{i=0}^{3}$ filters. Therefore, the delay is 4 units.

PROBLEM 5:
Problem 5.19 from the text P. P. Vaidyanathan (Multirate systems and filter banks). (4+3+3=10 points)
Solution:

(a) Given a 2 channel QMF filter bank. Analysis filters are given as $H_1(z) = H_0(-z)$. Synthesis filters $F_0(z) = H_0(z)$ and $F_1(Z) = -H_1(z)$.



The signals at various nodes of the figure are

$X_0(z) = H_0(z)X(z)$

$X_1(z) = H_1(z)X(z)$

$Y_0(z) = \frac{X_0(z)+X_0(-z)}{2}$

$Y_1(z) = \frac{X_1(z)+X_1(-z)}{2}$

$\hat{X}(z) = F_0(z)Y_0(z) + F_1(z)Y_1(z) = \frac{1}{2}\left[X_0(z)F_0(z) + X_1(z)F_1(z)\right] + \frac{1}{2}\left[X_0(-z)F_0(z) + X_1(-z)F_1(z)\right]$

$\implies \hat{X}(z) = \frac{1}{2}X(z)\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right] + \frac{1}{2}X(-z)\left[H_0(-z)F_0(z) + H_1(-z)F_1(z)\right]$

$\implies 2\hat{X}(z) = X(z)\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right] + X(-z)\left[H_0(-z)F_0(z) + H_1(-z)F_1(z)\right]$

Above can be written in matrix form as

$$2\hat{X}(z) = \begin{bmatrix} X(z) & X(-z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_0(-z) & H_0(z) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}$$

AC matrix $\boldsymbol{H}(z) = \begin{bmatrix} H_0(z) & H_0(-z) \\ H_0(-z) & H_0(z) \end{bmatrix}$.

det($\boldsymbol{H}(z)$)=$\left| \begin{bmatrix} H_0(z) & H_0(-z) \\ H_0(-z) & H_0(z) \end{bmatrix} \right| = H_0^2(z) - H_0^2(-z)$

(b) $T(z) = \frac{\hat{X}(z)}{X(z)} = \frac{1}{2}X(z)\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right] + \frac{1}{2}X(-z)\left[H_0(-z)F_0(z) + H_1(-z)F_1(z)\right]$

Now putting $F_0(z) = H_0(z)$ and $F_1(Z) = -H_1(z)$ in above equation,

$T(z) = \frac{1}{2}X(z)\left[H_0(z)H_0(z) - H_1(z)H_1(z)\right] = \frac{1}{2}X(z)\left[H_0(z)H_0(z) - H_0(-z)H_0(-z)\right]$

$\implies T(z) = \frac{1}{2}X(z)\left[H_0^2(z) - H_0^2(-z)\right] = \frac{1}{2}X(z) \times det(\boldsymbol{H}(z))$

Therefore $T(z)$ is 0 for some z if and only if det($\boldsymbol{H}(z)$)=0 (c)Given,

$H_0(z) = \sum_{n=0}^{N} h(n)z^{-n}$ and N is even.

$h(n)$ is real with $h(n) = h(N-n)$.

Taking Z-transform we get,

$H_0(z) = z^N H_0(z^{-1})$

Therefore, $H_0(-z) = (-z)^N H_0(-z^{-1}) = z^N H_0(-z^{-1})$ as N is even.

Taking $z = e^{\frac{j\pi}{2}}$ we get,

$H_0(-e^{\frac{j\pi}{2}}) = (e^{\frac{j\pi}{2}})^N H_0(-e^{-\frac{j\pi}{2}})$

$\implies H_0(-e^{\frac{j\pi}{2}}) = e^{\frac{jN\pi}{2}} H(e^{\frac{j\pi}{2}})$ as $H_0(-e^{-\frac{j\pi}{2}}) = H(e^{\frac{j\pi}{2}})$.

Now $\det(\boldsymbol{H}(z))$=$\left[H_0^2(z) - H_0^2(-z)\right]$

Therefore, $\det(\boldsymbol{H}(e^{\frac{j\pi}{2}}))$=$\left[H_0^2(e^{\frac{j\pi}{2}}) - H_0^2(-e^{\frac{j\pi}{2}})\right] = H_0^2(e^{\frac{j\pi}{2}}) - e^{\frac{jN\pi}{2}\times 2} H_0^2(-e^{\frac{j\pi}{2}})$

$\implies det(\boldsymbol{H}(e^{\frac{j\pi}{2}})) = H_0^2(e^{\frac{j\pi}{2}}) - e^{jN\pi} H_0^2(-e^{\frac{j\pi}{2}})$

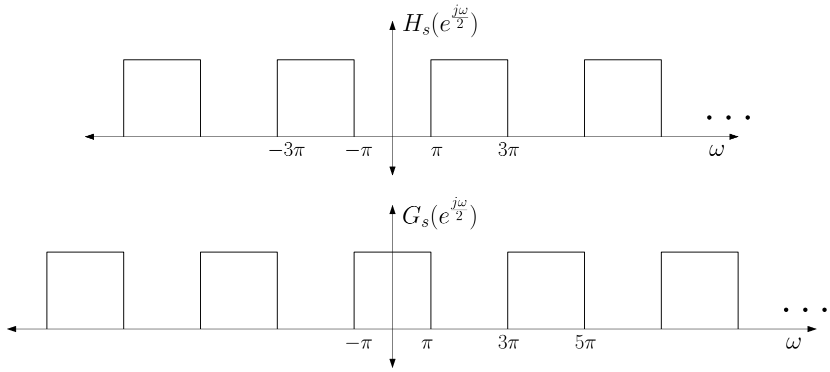
As N is even $e^{jN\pi} = 1$

$\implies det(\boldsymbol{H}(e^{\frac{j\pi}{2}})) = H_0^2(e^{\frac{j\pi}{2}}) - H_0^2(-e^{\frac{j\pi}{2}}) = 0$

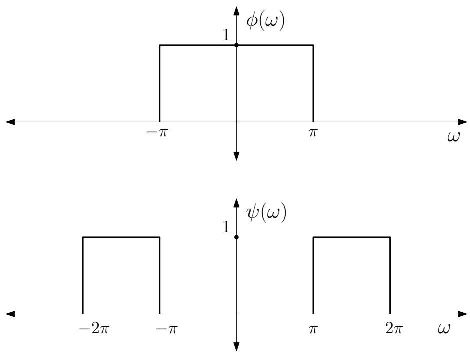
Therefore $\boldsymbol{H}(z)$ is singular for $z = e^{\frac{j\pi}{2}}$.

PROBLEM 6:
Problem 11.15 from the text P. P. Vaidyanathan (Multirate systems and filter banks). (5 points)
Solution:

Given $H_s(e^{\frac{j\omega}{2}})$ and $G_s(e^{\frac{j\omega}{2}})$



Given $\phi(\omega)$ and $\psi(\omega)$



Wavelet basis function $\psi_{kl}(t) = 2^{-\frac{k}{2}}\psi(2^{-k}t - l)$ as given in the question.

To prove that $\psi_{kl}(t)$ forms an orthonormal set, we have to show that,

$$\langle \psi_{kl_1}(t), \psi_{kl_2}(t) \rangle = \begin{cases} 0, & l_1 \neq l_2 \\ 1, & l_1 = l_2 \end{cases}$$

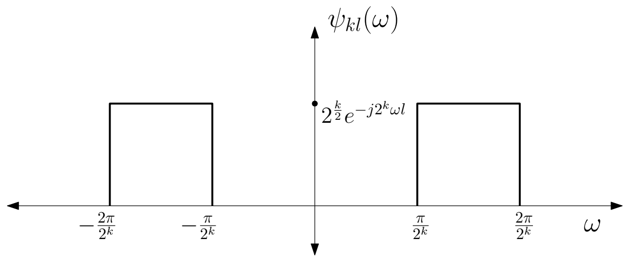
We have

$$\begin{aligned} \psi(t) &\longleftrightarrow \psi(\omega) \\ \psi(t-l) &\longleftrightarrow e^{-j\omega l}\psi(\omega) \\ \psi(2^{-k}t - l) &\longleftrightarrow 2^k e^{-j2^k\omega l}\psi(2^k\omega) \\ 2^{-\frac{k}{2}}\psi(2^{-k}t - l) &\longleftrightarrow 2^{\frac{k}{2}} e^{-j2^k\omega l}\psi(2^k\omega) \end{aligned}$$

Now, let's evaluate $\langle \psi_{kl_1}(t), \psi_{kl_2}(t) \rangle$.

$$\begin{aligned}
\langle \psi_{kl_1}(t), \psi_{kl_2}(t) \rangle &= \int_{-\infty}^{\infty} \psi_{kl_1}(t)\psi_{kl_2}(t)dt \\
&= \frac{1}{2\pi}[\psi_{kl_1}(\omega) \circledast \psi_{kl_2}(\omega)] \;\text{ at}\,\omega = 0 \\
&= \frac{1}{2\pi}\int_T \psi_{kl_1}(T)\psi_{kl_2}(\omega - T)dT \\
&= \frac{1}{2\pi}\int_T \psi_{kl_1}(T)\psi_{kl_2}(-T)dT \;\text{ since}\,\omega = 0
\end{aligned}$$

$\psi_{kl}(\omega)$ can be represented as,



Therefore,

$$\begin{aligned}
\langle \psi_{kl_1}(t), \psi_{kl_2}(t) \rangle &= \frac{1}{2\pi}\int_{-\frac{2\pi}{2^k}}^{-\frac{\pi}{2^k}} 2^{\frac{k}{2}}e^{-j2^kTl_1} \times 2^{\frac{k}{2}}e^{j2^kTl_2}dT + \frac{1}{2\pi}\int_{\frac{\pi}{2^k}}^{\frac{2\pi}{2^k}} 2^{\frac{k}{2}}e^{-j2^kTl_1} \times 2^{\frac{k}{2}}e^{j2^kTl_2}dT \\
&= \frac{1}{2\pi}\int_{-\frac{2\pi}{2^k}}^{-\frac{\pi}{2^k}} 2^k e^{-j2^kT(l_1-l_2)}dT + \frac{1}{2\pi}\int_{\frac{\pi}{2^k}}^{\frac{2\pi}{2^k}} 2^k e^{-j2^kT(l_1-l_2)}dT
\end{aligned}$$

Substituting $2^kT = p$ we get,

$$\langle \psi_{kl_1}(t), \psi_{kl_2}(t) \rangle = \frac{1}{2\pi}\int_{-2\pi}^{-\pi} e^{-jp(l_1-l_2)}dp + \frac{1}{2\pi}\int_{\pi}^{2\pi} e^{-jp(l_1-l_2)}dp$$

Substituting $p = q - 2\pi$ in the first integral and $p = q + 2\pi$ in the second and noting that $e^{-j(q-2\pi)(l_1-l_2)} = e^{-jq(l_1-l_2)}$ and $e^{-j(q+2\pi)(l_1-l_2)} = e^{-jq(l_1-l_2)}$, we get,

$$\begin{aligned}
\langle \psi_{kl_1}(t), \psi_{kl_2}(t) \rangle &= \frac{1}{2\pi}\int_{0}^{\pi} e^{-jq(l_1-l_2)}dq + \frac{1}{2\pi}\int_{-\pi}^{0} e^{-jq(l_1-l_2)}dq \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jq(l_1-l_2)}dq \\
&= \delta(l_2 - l_1) \\
&= \begin{cases} 1, & \text{for}\, l_1 = l_2 \\ 0, & \text{for}\, l_1 \neq l_2 \end{cases}
\end{aligned}$$

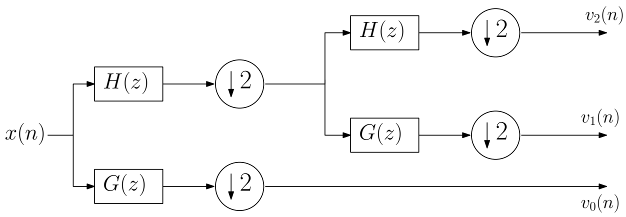
Therefore $\psi_{kl}(t)$ forms an orthonormal set.

PROBLEM 7:
(a) Represent the Haar wavelet decomposition and reconstruction upto second scale as non uniform filter bank (i.e., decimation and upsampling rates are non uniform across different channels). What are the analysis filters $H_i(z)$ and synthesis filters $F_i(z)$ for this filter bank. (4 points)
(b) Using the multirate theory in the frequency domain, show that this filter bank achieves perfect reconstruction. (3 points)
(c) Test which of the special properties given below are satisfied by the filter bank. (2 points)

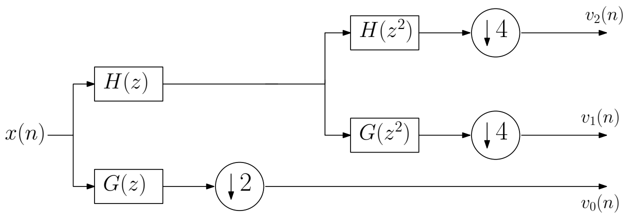
1. Strictly complementary
2. Power complementary
3. All pass complementary
4. Doubly complementary

(d) Are they Nyquist-m? (1 point)

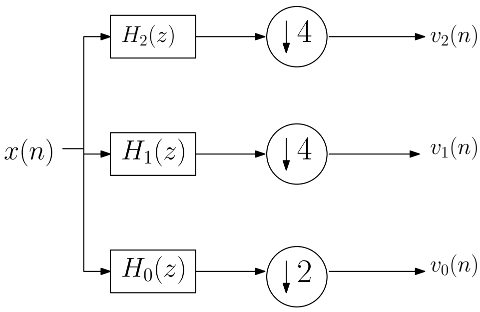
Solution:
We can simplify the structure of Haar wavelet and decomposition as



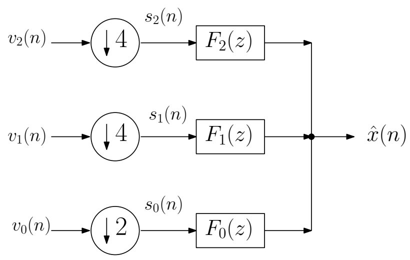
Using Nobel identities



This simplifies the analysis side to



Similarly the synthesis part can be simplified to



(a)We have for the analysis side,

$$
\begin{aligned}
H(z) &= \frac{1+z^{-1}}{\sqrt{2}}; G(z) = \frac{1-z^{-1}}{\sqrt{2}} \\
H_2(z) &= H(z)H(z^2) = \frac{1+z^{-1}+z^{-2}+z^{-3}}{2} \\
H_1(z) &= H(z)G(z^2) = \frac{1+z^{-1}-z^{-2}-z^{-3}}{2} \\
H_0(z) &= G(z) = \frac{1-z^{-1}}{\sqrt{2}}
\end{aligned}
$$

Similarly for the synthesis side,

$$
\begin{aligned}
H_s(z) &= \frac{1+z}{\sqrt{2}}; G_s(z) = \frac{1-z}{\sqrt{2}} \\
F_2(z) &= H_s(z)H_s(z^2) = \frac{1+z+z^2+z^3}{2} \\
F_1(z) &= H_s(z)G_s(z^2) = \frac{1+z-z^2-z^3}{2} \\
F_0(z) &= G_s(z) = \frac{1-z}{\sqrt{2}}
\end{aligned}
$$

(b)We have $\omega_4 = e^{-j2\pi/4} = e^{-j\pi/2}; \omega_2 = e^{-j2\pi/2} = e^{-j\pi}$.

$$
\begin{aligned}
V_2(z) &= \frac{1}{4}\sum_{k=0}^{3} X(z^{\frac{1}{4}}\omega_4^k)H_2(z^{\frac{1}{4}}\omega_4^k) \\
V_1(z) &= \frac{1}{4}\sum_{k=0}^{3} X(z^{\frac{1}{4}}\omega_4^k)H_1(z^{\frac{1}{4}}\omega_4^k) \\
V_0(z) &= \frac{1}{2}\sum_{k=0}^{1} X(z^{\frac{1}{2}}\omega_2^k)H_0(z^{\frac{1}{2}}\omega_2^k)
\end{aligned}
$$

We can write expressions for $S_2(z)$, $S_1(z)$ and $S_0(z)$ as,

$$S_2(z) = \frac{1}{4}\sum_{k=0}^{3} X(z\omega_4^k)H_2(z\omega_4^k)$$
$$S_1(z) = \frac{1}{4}\sum_{k=0}^{3} X(z\omega_4^k)H_1(z\omega_4^k)$$
$$S_0(z) = \frac{1}{2}\sum_{k=0}^{1} X(z\omega_2^k)H_0(z\omega_2^k)$$

where,

$$\omega_4^k = \begin{cases} 1 & k=0 \\ j & k=1 \\ -1 & k=2 \\ -j & k=3 \end{cases}$$

$$\omega_4^k = \begin{cases} 1 & k=0 \\ -1 & k=1 \end{cases}$$

We know

$$\begin{aligned} \hat{X}(z) &= S_2(z)F_2(z) + S_1(z)F_1(z) + S_0(z)F_0(z) \\ &= \frac{F_2(z)}{4}[X(z)H_2(z) + X(jz)H_2(jz) + X(-z)H_2(-z) + X(-jz)H_2(-jz)] \\ &+ \frac{F_1(z)}{4}[X(z)H_1(z) + X(jz)H_1(jz) + X(-z)H_1(-z) + X(-jz)H_1(-jz)] \\ &+ \frac{F_0(z)}{2}[X(z)H_0(z) + X(-z)H_0(-z)] \end{aligned}$$

This can be rearranged as,

$$\begin{aligned} \hat{X}(z) = X(z)&\left[\frac{F_2(z)H_2(z)}{4} + \frac{F_1(z)H_1(z)}{4} + \frac{F_0(z)H_0(z)}{2}\right] \\ +\ X(jz)&\left[\frac{F_2(z)H_2(jz)}{4} + \frac{F_1(z)H_1(jz)}{4}\right] \\ +\ X(-z)&\left[\frac{F_2(z)H_2(-z)}{4} + \frac{F_1(z)H_1(-z)}{4} + \frac{F_0(z)H_0(-z)}{2}\right] \\ +\ X(-jz)&\left[\frac{F_2(z)H_2(-jz)}{4} + \frac{F_1(z)H_1(-jz)}{4}\right] \end{aligned}$$

Plugging in the values of the analysis and synthesis filters we find that

$$\left[\frac{F_2(z)H_2(z)}{4} + \frac{F_1(z)H_1(z)}{4} + \frac{F_0(z)H_0(z)}{2}\right] = 1$$

$$\left[\frac{F_2(z)H_2(jz)}{4} + \frac{F_1(z)H_1(jz)}{4}\right] = 0$$

$$\left[\frac{F_2(z)H_2(-z)}{4} + \frac{F_1(z)H_1(-z)}{4} + \frac{F_0(z)H_0(-z)}{2}\right] = 0$$

$$\left[\frac{F_2(z)H_2(-jz)}{4} + \frac{F_1(z)H_1(-jz)}{4}\right] = 0$$

Thus we find that,

$$\hat{X}(z) = X(z)$$

which shows that we achieve perfect reconstruction.

(c) (i) Strictly complementary: We observe that

$$\sum_{i=0}^{2} H_i(z) = 1 + \frac{1}{\sqrt{2}} + z^{-1}\left[1 - \frac{1}{\sqrt{2}}\right]$$

$$\sum_{i=0}^{2} F_i(z) = 1 + \frac{1}{\sqrt{2}} + z\left[1 - \frac{1}{\sqrt{2}}\right]$$

So the filters are not strictly complementary as their summation is not a pure delay.

(ii) All pass complementary:

$\sum_{i=0}^{3} H_i(z)$ and $\sum_{i=0}^{3} F_i(z)$ are not all pass functions. So they filters are not all pass complementary.

(iii) Power complementary: We observe that

$$\begin{aligned}\sum_{i=0}^{2} |H_i(z)|^2 &= H_0(z)H_0^*(z) + H_1(z)H_1^*(z) + H_2(z)H_2^*(z)\\ &\neq \text{constant}\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{2} |F_i(z)|^2 &= F_0(z)F_0^*(z) + F_1(z)F_1^*(z) + F_2(z)F_2^*(z)\\ &\neq \text{constant}\end{aligned}$$

Therefore, analysis and synthesis filters are not power complimentary.

(iv) Doubly complementary: Analysis and synthesis filters are neither power nor all-pass complimentary. So they are not doubly complimentary.

(d) Filters are Nyquist-m if they can be expressed in the form $A_i(z) = c + z^{-1}E(z^2)$. As the filters cannot be expressed in this form, they are not Nyquist-m.