

Homework #1 solutions

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Linear and non-linear programming-1
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Problem 2.1.

Solution. (a) Introducing slack and surplus variables, we get

$$\begin{aligned} \text{minimize} \quad & x + 2y + z \\ \text{subject to} \quad & x + y - \alpha = 2 \\ & x + y + \beta = 3 \\ & x + z - \gamma = 4 \\ & x + z + \delta = 5 \\ & \alpha, \beta, \gamma, \delta \geq 0 \\ & x, y, z \geq 0. \end{aligned}$$

(b) Introducing slack and surplus variables, we get

$$\begin{aligned} \text{minimize} \quad & \alpha + \beta + \gamma + 4 \\ \text{subject to} \quad & \alpha + 2\beta + 3\gamma = 2 \\ & \alpha, \beta, \gamma \geq 0. \end{aligned}$$

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Problem 2.2.

Solution. (a) From Figure 1a, we see that the cost is given by

$$\begin{aligned} \text{Cost} = & (x_{1A} + 1.5x_{1B} + 2x_{2A} + 1.5x_{2B}) \\ & + (y_{Am_1} + 2y_{Am_2} + y_{Am_3} + 3y_{Bm_1} + 4y_{Bm_2} + 2y_{Bm_3}) \end{aligned} \quad (1)$$

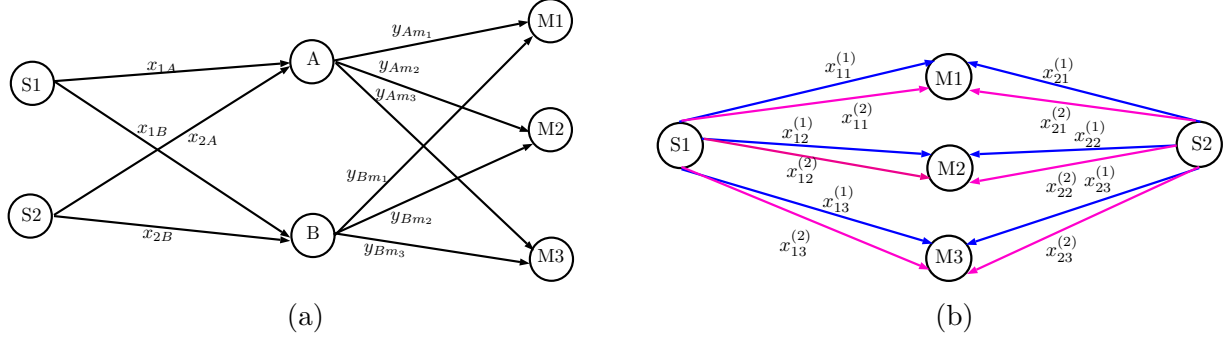


Figure 1

minimize Cost (Equation 1)

$$\begin{aligned}
 \text{subject to } x_{1A} + x_{1B} &= 10 \\
 x_{2A} + x_{2B} &= 15 \\
 y_{Am_1} + y_{Bm_1} &= 8 \\
 y_{Am_2} + y_{Bm_2} &= 14 \\
 y_{Am_3} + y_{Bm_3} &= 3 \\
 x_{1A}, x_{1B}, x_{2A}, x_{2B} &\geq 0 \\
 y_{Am_1}, y_{Am_2}, y_{Am_3} &\geq 0 \\
 y_{Bm_1}, y_{Bm_2}, y_{Bm_3} &\geq 0.
 \end{aligned}$$

(b) From Figure 1b, we see that the cost is given by

$$\begin{aligned}
 \text{Cost} &= \left(5x_{11}^{(1)} + 3.5x_{11}^{(2)} + 3x_{12}^{(1)} + 5.5x_{12}^{(2)} + 2x_{13}^{(1)} + 3.5x_{13}^{(2)} \right) \\
 &+ \left(6x_{21}^{(1)} + 4.5x_{21}^{(2)} + 3x_{22}^{(1)} + 3.5x_{22}^{(2)} + 2x_{23}^{(1)} + 3.5x_{23}^{(2)} \right) \quad (2)
 \end{aligned}$$

minimize Cost (Equation 2)

$$\begin{aligned}
 \text{subject to } \sum_{i=1}^2 x_{1k}^{(i)} + x_{2k}^{(i)} &= \alpha_k \quad \forall k = 1, 2, 3 \\
 \sum_{k=1}^3 x_{1k}^{(1)} + x_{1k}^{(2)} &= 10 \\
 \sum_{k=1}^3 x_{2k}^{(1)} + x_{2k}^{(2)} &= 15 \\
 x_{1k}^{(1)}, x_{2k}^{(1)} &\geq 0 \quad \forall k = 1, 2, 3.
 \end{aligned}$$

where

$$\alpha_k = \begin{cases} 8 & k = 1 \\ 14 & k = 2 \\ 3 & k = 3 \end{cases}$$

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Problem 2.8.

Solution. Let us assume the following: $|x| \leq t_1$, $|y| \leq t_2$, $|z| \leq t_3$ for some t_1, t_2 , and $t_3 \geq 0$. This results in the following inequalities.

$$x \leq t_1 \tag{3}$$

$$x \geq -t_1 \tag{4}$$

$$y \leq t_2 \tag{5}$$

$$y \geq -t_2 \tag{6}$$

$$z \leq t_3 \tag{7}$$

$$z \geq -t_3 \tag{8}$$

Introducing slack and surplus variables we get

$$2x + z = 3 \tag{9}$$

$$x + y + \alpha_1 = 1 \tag{10}$$

$$x + \alpha_2 = t_1 \tag{11}$$

$$x - \alpha_3 = -t_1 \tag{12}$$

$$y + \alpha_4 = t_2 \tag{13}$$

$$y - \alpha_5 = -t_2 \tag{14}$$

$$z + \alpha_6 = t_3 \tag{15}$$

$$z - \alpha_7 = -t_3 \tag{16}$$

From equations (9)-(16), we get

$$2t_1 - \alpha_2 - \alpha_3 = 0 \tag{17}$$

$$2t_2 - \alpha_4 - \alpha_5 = 0 \tag{18}$$

$$2t_3 - \alpha_6 - \alpha_7 = 0 \tag{19}$$

$$t_1 + t_2 - \alpha_1 - \alpha_3 - \alpha_5 = -1 \tag{20}$$

$$2t_1 + t_3 - \alpha_1 - 2\alpha_3 = -3 \tag{21}$$

The standard form the problem is given by

$$\text{minimize } t_1 + t_2 + t_3$$

$$\text{subject to } t_1 + t_2 - \alpha_1 - \alpha_3 - \alpha_5 = -1$$

$$2t_1 + t_3 - \alpha_1 - 2\alpha_3 = -3$$

$$2t_1 - \alpha_2 - \alpha_3 = 0$$

$$2t_2 - \alpha_4 - \alpha_5 = 0$$

$$2t_3 - \alpha_6 - \alpha_7 = 0$$

$$t_i, \alpha_j \geq 0 \quad \forall i = 1, 2, 3 \text{ and } j = 1, \dots, 7.$$

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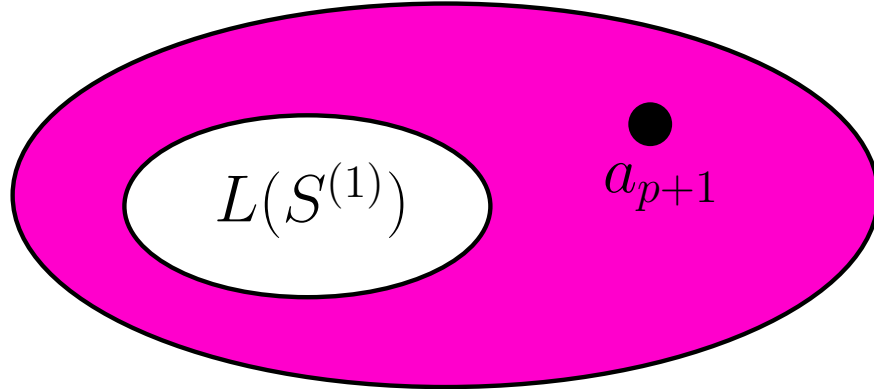


Figure 2

Problem 2.12.

Solution. Let $S^{(1)} = \{a_1, \dots, a_p\}$ and $L(S^{(1)}) = \text{span}\{a_1, \dots, a_p\}$ denote the span of vectors as shown in Figure 2. Consider a vector $a_{p+1} \notin L(S^{(1)})$. Now, equating linear combination of $\{a_1, \dots, a_{p+1}\}$ to zero we get

$$\alpha_1 a_1 + \dots + \alpha_p a_p + \alpha_{p+1} a_{p+1} = 0. \quad (22)$$

From equation (22) we see that $a_{p+1} = 0$ (because the set $S^{(1)}$ is a linearly independent set). Now, the set $S^{(2)} = \{a_1, \dots, a_p, a_{p+1}\}$ forms a linearly independent set. Similarly consider another vector $a_{p+2} \notin L(S^{(2)})$ and we see that set $S^{(3)} = \{a_1, \dots, a_{p+1}, a_{p+2}\}$ forms a linearly independent set. This can be repeated till we find $m - p$ vectors. ■

Problem 2.3.

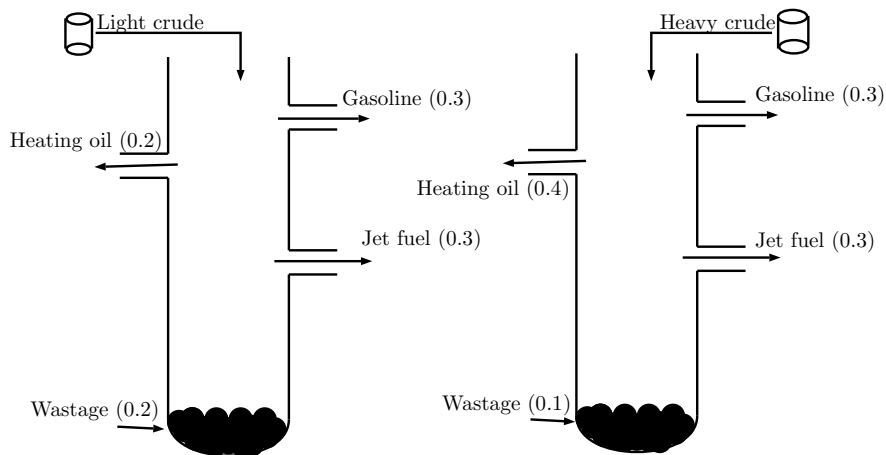


Figure 3

Solution. From the table we see that, 1 barrel of light crude produces 0.3 barrels of gasoline, 0.2 barrels of heating oil, and 0.3 barrels of jet fuel. Similarly, 1 barrel of heavy crude

produces 0.3 barrels of gasoline, 0.4 barrels of heating oil, and 0.2 barrels of jet fuel. Let x_l and x_h be the barrels of light crude and heavy crude respectively. Therefore, the cost is

$$C = 35x_l + 30x_h \tag{23}$$

The standard form of the problem is

$$\begin{aligned} &\text{minimize} && C \\ &\text{subject to} && 0.3x_l + 0.3x_h = 900000 \\ & && 0.2x_l + 0.4x_h = 800000 \\ & && 0.3x_l + 0.2x_h = 500000 \\ & && x_l, x_h \geq 0. \end{aligned}$$

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Problem 2.9.

Solution. Let $y = \text{maximum}(c_1^T x + d_1, \dots, c_p^T x + d_p)$. Then we have to minimize y subject to the following constraints:

$$\begin{aligned} &\text{minimize} && y \\ &\text{subject to} && c_i^T x + d_i \leq y \\ & && c_i, d_i \geq 0 \quad \forall i = 1, \dots, p. \end{aligned}$$

Introducing slack variables we get

$$\begin{aligned} &\text{minimize} && y \\ &\text{subject to} && c_i^T x + d_i + \alpha_i = y \\ & && c_i, d_i, \alpha_i \geq 0 \quad \forall i = 1, \dots, p. \end{aligned}$$

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