Indian Institute of Science

Linear and non-linear programming-1

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Home Work #3, Spring 2018

Late submission policy: Points scored = Correct points scored $\times e^{-d}$, d = # days late

Assigned date: Mar. 16^{th} 2018 **Due date:** Mar. 25^{th} 2018 in class

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PROBLEM 1: Consider the linear programming problem:

$$\begin{aligned} \text{minimize} \quad & x_1 - x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 \\ & 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 \\ & -x_1 - x_2 + 2x_3 + x_4 = 6 \\ & x_1 \leq 0 \\ & x_2, x_3 \geq 0 \end{aligned}$$

write down the corresponding dual problem. (10 pts.)

PROBLEM 2: The purpose of this exercise is to show that solving linear programming problem is no harder than solving system of linear equalities. Suppose that we are given a subroutine which, given a system of linear inequality constraints, either produces a solution or decides that no solution exist. Construct a simple algorithm that uses a single call to this subroutine and which finds an optimal solution to any linear programming problem that has an optimal solution. (10 pts.)

PROBLEM 3: Consider a linear programming problem in standard form and assume that the rows of A are linearly independent. For each one of the following statements, provide either a proof or a counterexample.

(a) Let $x^\star$ be a basic feasible solution. Suppose that for every basis corresponding to $x^\star$, the associated basic solution to the dual is infeasible. Then, optimal cost must be strictly less than $c^{\mathrm{T}} x^\star$.
(b) The dual of the axillary primal problem considered in phase I of simplex method is always feasible.
(c) Let p_i be the dual variable associated with the i^{th} equality constraint in the primal. Eliminating the i^{th} primal equality constraint is equivalent to introducing the additional constraint $p_i = 0$ in the dual problem.
(d) If the unboundedness criterion in the primal simplex algorithm is satisfied, then the dual problem is infeasible.

(10 pts.)

PROBLEM 4: Consider the problem of minimizing $\max_{i=1,\ldots,m} \left(a_i^{\mathrm{T}} x - b_i\right)$ over all $x \in \Re^{\mathrm{n}}$. Let v be the value of the optimal cost, assumed finite. Let A be the matrix with rows $a_1, \ldots, a_m$, and let b the vector with components $b_1, \ldots, b_m$.

(a) Consider any vector $p \in \Re^{\mathrm{m}}$ that satisfies $p^{\mathrm{T}} A = 0^{\mathrm{T}}$, $p \geq 0$, and $\sum_{i=1}^{m} p_i = 1$. Show that $-p^{\mathrm{T}} b \leq v$.
(b) In order to obtain the best possible lower bound of the form considered in part (a), we form the linear programming problem

$$\begin{aligned} \text{maximize} \quad & -p^{\mathrm{T}} b \\ \text{subject to} \quad & p^{\mathrm{T}} A = 0^{\mathrm{T}} \\ & p^{\mathrm{T}} e = 1 \\ & p \geq 0 \end{aligned}$$

where e is the vector with all components equal to 1. Show that the optimal cost in this problem is equal to v.

(10 pts.)

PROBLEM 5: Let A be a given matrix. Show that exactly one of the following alternatives must hold.

(a) There exists some $x \neq 0$ such that $Ax = 0, x \geq 0$

(b) There exists some p such that $p^{\mathrm{T}} A > 0^{\mathrm{T}}$

(10 pts.)

PROBLEM 6: Let $f : \Re^{\mathrm{n}} \to \Re$ be a convex function and let $S \subset \Re^{\mathrm{n}}$ be a convex set. Let $x^\star$ be an element of S. Suppose that $x^\star$ is a local optimum for the problem of minimizing $f(x)$ over S; that is, there exists some $\epsilon > 0$ such that $f(x^\star) \leq f(x)$ for all $x \in S$ for which $\|x - x^\star\| \leq \epsilon$. Prove that $x^\star$ is globally optimal; that is $f(x^\star) \leq f(x)$ for all $x \in S$. (10 pts.)

PROBLEM 7: Consider the problem of minimizing $C^{\mathrm{T}} x$ over a polyhedron P. Prove the following:

(a) A feasible solution x is optimal if and only if $C^{\mathrm{T}} d \geq 0$ for every feasible direction d at x.

(b) A feasible solution x is the unique optimal solution if and only if $C^{\mathrm{T}} d > 0$ for every nonzero feasible direction d at x.

(10 pts.)

PROBLEM 8: Let x be an element of the standard form polyhedron $P = \{x \in \Re^{\mathrm{n}} | Ax = b, x \geq 0\}$. Prove that a vector $d \in \Re^{\mathrm{n}}$ is a feasible direction at x if and only if $Ad = 0$ and $d_i \geq 0$ for every i such that $x_i = 0$. (10 pts.)

PROBLEM 9: Let $P = \{x \in \Re^3 | x_1 + x_2 + x_3 = 1, x \geq 0\}$ and consider the vector $x = (0, 0, 1)$. Find the set of feasible directions at x. (10 pts.)

PROBLEM 10: Let x be a basic feasible solution associated with some basis matrix B. Prove the following:

(a) If the reduced cost of every nonbasic variable is positive, then x is the unique optimal solution.

(b) If x is the unique optimal solution and is non degenerate, the the reduced cost of every nonbasic variable is positive.

(10 pts.)