Indian Institute of Science
E9−207: Basics of Signal Processing
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Homework #2 Solutions, Spring 2018
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Late submission policy: Points scored = Correct points scored $\times e^{-d}$, $d = \#$ days late
Assigned date: Feb. 20th 2018 **Due date**: Mar. 1st 2018 by end of the day

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PROBLEM 1:
(a)Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two vector spaces. Then show that $\mathcal{S}_1 \cap \mathcal{S}_2$ is also a vector space.
(b)If A $\in \mathbb{C}^{n\times n}$ and $u, v \in \mathbb{C}^n$ are non-zero vectors such that $Au = 2u$ and $Av = 3v$, show that u, v are linearly independent.

Solution:
(a) Consider two vectors x, y such that $x, y \in \mathcal{S}_1$ and $x, y \in \mathcal{S}_2$. This implies $x, y \in \mathcal{S}_1 \cap \mathcal{S}_2$. Similarly $\alpha x + \beta y \in \mathcal{S}_1$ and $\alpha x + \beta y \in \mathcal{S}_2$ for scalars α, β. Therefore, $\alpha x + \beta y \in \mathcal{S}_1 \cap \mathcal{S}_2$. The zero vector, $\underset{\sim}{0} \in \mathcal{S}_1$ and $\underset{\sim}{0} \in \mathcal{S}_2$. Therefore, $\underset{\sim}{0} \in \mathcal{S}_1 \cap \mathcal{S}_2$. Hence, $\mathcal{S}_1 \cap \mathcal{S}_2$ is a vector space.
(b)Suppose u, v are linearly dependent. Then we can write $u = kv$ for some scalar $k \neq 0$. Then, $Au = A(kv) = kAv = 3kv$. But $Au = 2u$. This means $2u = 3kv$ or $u = \frac{3k}{2}v$.
Now, $u = kv$ and $u = \frac{3k}{2}v$ both hold if only if $k = 0$. This is a contradiction as $k \neq 0$ as per assumption. Therefore, u and v are linearly independent.

PROBLEM 2:
(a)Let $A \in \mathbb{C}^{m\times m}$ be a matrix acting on vectors in the vector space $\mathbb{C}^m$. We define a new product between vectors $x, y \in \mathbb{C}^m$ as $(x, y)_A$ as $x^\dagger Ay$. Under what conditions is this a valid inner product?
(b)Consider the matrix

$$A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$$

For what values of $a \in \mathbb{C}$ is $\sqrt{x\dagger Ax}$ a norm defined on $\mathbb{C}^3$?
Note: $a^\dagger$ is the transpose conjugate of a. For example

$$v = \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix} \Rightarrow v^\dagger = \begin{pmatrix} 1 & -i & i \end{pmatrix}$$

Solution:
(a) To be a valid inner product we need $\langle x, x\rangle_A \geq 0$ i.e., $x^\dagger Ax \geq 0$. This should hold for all x. This holds if and only if A is positive definite matrix.
(b) A is positive semi-definite implies $A^\dagger = A \Rightarrow a \in \mathbb{R}$. Secondly all its eigen values must be positive. Therefore $\lambda = 1 - a, 1 - a, 1 + 2a$ must be positive. Therefore $a \in (-\frac{1}{2}, 1)$.

PROBLEM 3:
What is the minimum value of $x - y - z$ subject to the constraint $x^2 + y^2 + z^2 = 1$?
Solution:
For two vectors u, v we have $|\langle u, v\rangle| \leq ||u||\ ||v||$. Fix $u = (x, y, z)^T, v = (1, -1, -1)^T$. Then $||u|| = \sqrt{x^2 + y^2 + z^2}, ||v|| = \sqrt{1^2 + (-1)^2 + (-1)^2}$.
$\therefore |\langle u, v\rangle| = |x - y - z| \leq \sqrt{x^2 + y^2 + z^2} \cdot \sqrt{1^2 + (-1)^2 + (-1)^2} = 1 \cdot \sqrt{3}$.
$\Rightarrow -\sqrt{3} \leq x - y - z \leq \sqrt{3}$. This implies that the minimum value of $x - y - z$ is $-\sqrt{3}$.

PROBLEM 4:

Consider the functions $\varphi_k(t) = A\,\text{sinc}(\pi(t-k))$ where k is an integer and $A \in \mathbb{C}$.
For integers k, l evaluate

$$\int_R \varphi_k(t)\varphi_l^*(t)dt$$

Conclude that $\varphi(t) \in L^2(\mathbb{R})$ and that $\{\varphi_k : k \in \mathbb{Z}\}$ forms an orthonormal set of functions in $L^2(\mathbb{R})$.

Solution:

Denote

$$\psi_{k,l}(t) = \varphi_k(t)\varphi_l^*(t)dt$$
$$\hat{\psi}_{k,l}(\omega) = \int_R \psi_{k,l}(t)e^{-j\omega t}dt$$

We need to evaluate

$$\hat{\psi}_{k,l}(\omega)|_{\omega=0} = \int_R \varphi_k(t)\varphi_l^*(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\varphi}_k(\omega)\hat{\varphi}_l^*(-\omega)d\omega|_{\omega=0}$$

We know

$$\phi_k(t) = A\,\text{sinc}(\pi(t-k)) \longleftrightarrow A\,\text{rect}\left(\frac{\omega}{2\pi}\right)e^{-j\omega k}$$
$$\phi_l^*(t) = A^*\,\text{sinc}(\pi(t-l)) \longleftrightarrow A^*\,\text{rect}\left(\frac{\omega}{2\pi}\right)e^{j\omega l}$$

Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\varphi}_k(\omega)\hat{\varphi}_l^*(-\omega)d\omega|_{\omega=0}$$
$$= \frac{1}{2\pi}|A|^2\int_{-\pi}^{\pi} e^{-j\omega(k-l)}d\omega|_{\omega=0}$$
$$= |A|^2\frac{\sin\pi(k-l)}{\pi(k-l)} = \begin{cases} 0 \text{ for } l \neq k \\ |A|^2 \text{ for } l = k \end{cases}$$

Therefore,

$$\int_R \varphi_k(t)\varphi_k^*(t)dt = \int_R |\varphi_k(t)|^2 dt = |A|^2 < \infty.$$

$|A|^2 = 1$ then $\{\varphi_k : k \in \mathbb{Z}\}$ form an orthonormal set of functions.

PROBLEM 5:

(a)A baseband signal $s(t)$ with 50 Hz bandwidth is sampled at a rate Fs. The resultant signal is downsampled by a factor 2 to obtain the discrete samples $\hat{s}(n)$. What is the minimum value of F_s in Hz to reconstruct back the signal $s(t)$ from the samples $\hat{s}(n)$?
(b) Let $s(n)$ be any discrete time signal with energy E_s. The signal is downsampled by 3. What is the evergy of the resultant signal if there is no aliasing after decimation?

Solution:

(a) $\frac{Fs}{2} \geq 100\text{Hz}$. Therefore, minimum value of $F_s = 200\text{Hz}$
(b) Let $X(\omega)$ be the frequency response of the original signal. The frequency response after

downsampling is

$$\hat{X}(z) = \frac{1}{3}\left(X\left(1 \cdot z^{\frac{1}{3}}\right) + X\left(\Omega \cdot z^{\frac{1}{3}}\right) + X\left(\Omega^2 \cdot z^{\frac{1}{3}}\right)\right) \text{ where } \Omega^3 = 1,$$

$$\Longrightarrow \hat{X}(\omega) = \frac{1}{3}\left(X\left(\frac{\omega}{3}\right) + X\left(\frac{\omega - 2\pi}{3}\right) + X\left(\frac{\omega - 4\pi}{3}\right)\right).$$

Since there is no aliasing, the responses $X\left(\frac{\omega}{3}\right)$, $X\left(\frac{\omega-2\pi}{3}\right)$ and $X\left(\frac{\omega-4\pi}{3}\right)$ do not overlap i.e., $X\left(\frac{\omega}{3}\right)X\left(\frac{\omega-2\pi}{3}\right)X\left(\frac{\omega-4\pi}{2}\right) = 0\,\forall\omega$.

Therefore,

$$\int_0^{2\pi}\left|\hat{X}(\omega)\right|^2 d\omega = \frac{1}{3}\int_0^{6\pi}\left|\hat{X}(\omega)\right|^2 d\omega = \frac{1}{27}\int_0^{6\pi}\left|X\left(\frac{\omega}{3}\right)\right|^2 d\omega + \frac{1}{27}\int_0^{6\pi}\left|X\left(\frac{\omega-2\pi}{3}\right)\right|^2 d\omega + \frac{1}{27}\int_0^{6\pi}\left|X\left(\frac{\omega-4\pi}{3}\right)\right|^2 d\omega$$

$$= \frac{1}{9}\int_0^{2\pi}|X(\omega_1)|^2 d\omega_1 + \frac{1}{9}\int_{-\pi}^{\pi}|X(\omega_2)|^2 d\omega_2 + +\frac{1}{9}\int_{-\pi}^{\pi}|X(\omega_3)|^2 d\omega_2$$

$$= \frac{1}{9}E_s + \frac{1}{9}E_s + \frac{1}{9}E_s = \frac{1}{3}E_s.$$

We have substituted $\omega_1 = \frac{\omega}{3}$, $\omega_2 = \frac{\omega-2\pi}{3}$ and $\omega_3 = \frac{\omega-4\pi}{3}$ in the above equations.

PROBLEM 6:
(a) A signal $x(t)$ is obtained by convolving signals $x_1(t)$ and $x_2(t)$ with the following characteristics:

$$|X_1(\omega)| = 0 \text{ for } |\omega| > 500\pi,$$
$$|X_2(\omega)| = 0 \text{ for } |\omega| > 250\pi.$$

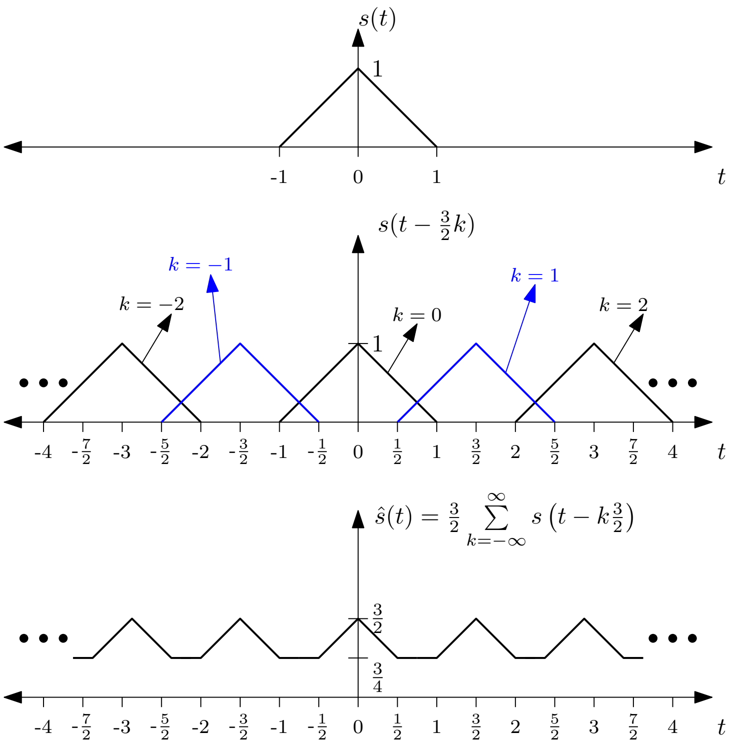
Impulse train sampling is performed on $x(t)$ to get $x_s(t) = \sum_{-\infty}^{\infty} x(nT)\delta(t - nT)$. Specify the range of values of T so that $x(t)$ may be recovered from $x_s(t)$. (4 pts)

(b) The signal $s(t) = \begin{cases} 1 - |t| \text{ for } -1 \le t \le 1 \\ 0 \text{ otherwise.} \end{cases}$ is passed through a system to obtain the output $\hat{s}(t)$. The system has a resonant frequency of $\frac{2}{3}$ Hz and hence allows only frequencies of $\frac{2}{3}$ Hz and its harmonics along with d.c. component. What is the value of $\int_{-2}^{2}|\hat{s}(t)|^2$? (8 pts)
Solution:
(a) Convolution in time domain is equivalent to multiplication it the frequency domain.
$X(\omega) = 0$ for $|\omega| > 250\pi \Rightarrow X(f) = 0$ for $|f| > 125$ Hz. Sampling frequency should be $f_s \geq 2 \cdot 125$ Hz. That is, the sampling period should be $T < \frac{1}{250}s = 4$ms.

(b) This is same as sampling in the frequency domain. Here, the frequency response $S(f)$ is multiplied by $\sum_{k=-\infty}^{\infty}\delta\left(f - k\frac{2}{3}\right)$. Therefore, it results in time-domain signal convolved with its Fourier inverse $\sum_{k=-\infty}^{\infty}e^{-j2\pi t\frac{2}{3}k} = \frac{3}{2}\sum_{k=-\infty}^{\infty}\delta\left(t - k\frac{3}{2}\right)$. Therefore the output signal is

$$\hat{s}(t) = s(t) * \frac{3}{2}\sum_{k=-\infty}^{\infty}\delta\left(t - k\frac{3}{2}\right) = \frac{3}{2}\sum_{k=-\infty}^{\infty}s\left(t - k\frac{3}{2}\right).$$

The input and output signals are shown in the following figure:



From the figure, the energy of $\hat{s}(t)$ in the interval $[-2, 2]$ is

$$\int_{-2}^{2} |\hat{s}(t)|^2\, dt = 3 \underbrace{\int_{-0.5}^{0.5} |\hat{s}(t)|^2\, dt}_{\text{3 triangle portions}} + 2 \underbrace{\int_{0.5}^{1} |\hat{s}(t)|^2\, dt}_{\text{2 flat portions}}$$

$$= 6 \int_{0}^{0.5} |\hat{s}(t)|^2\, dt + 2 \int_{0.5}^{1} |\hat{s}(t)|^2\, dt$$

$$= 6 \int_{0}^{0.5} \left| \frac{3}{2}(1-t) \right|^2 dt + 2 \int_{0.5}^{1} \left| \frac{3}{4} \right|^2 dt$$

$$= \frac{27}{2} \left[-\frac{(1-t)^3}{3} \right]_0^{0.5} + \frac{9}{16} = \frac{27}{2} \times \left(-\frac{1}{8 \times 3} + \frac{1}{3} \right) + \frac{9}{16} = \frac{63}{16} + \frac{9}{16} = \frac{9}{2}.$$

Therefore, $\int_{-2}^{2} |\hat{s}(t)|^2\, dt = \frac{18}{4} = 4.5$.