Indian Institute of Science
E9−207: Basics of Signal Processing
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Homework #1 Solutions, Spring 2018
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PROBLEM 1:
Give examples (if any) of 2D discrete time systems that are (a) non-causal but stable (b) non-causal and unstable. (4 pts.)

Solution:

(a) Non-causal but stable:

$$y(n_1, n_2) = x(n_1 + k_1, n_2 + k_2)$$

n_1 and n_2 are time parameters in 2 dimensions and k_1 and k_2 are positive integers. The system is non-causal because output depends on future values of input. The system is stable because output is bounded if input is bounded.

(b) Non-causal and unstable:

$$y(n_1, n_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} x(n_1 + k_1, n_2 + k_2)$$

The system is non-causal because output depends on future values of input. The system is unstable because output is unbounded for bounded input.

PROBLEM 2:
Find the modes of a system whose impulse response is $y[n] = n^k u[n]$, where k is a positive integer. Write the governing difference equation for this system. (6 pts.)

Solution:
The impulse response is given as $y[n] = n^k u[n]$.

Let Y(z) denote the z-transform of the system.

The Z-transform of a unit step function is,

$$\sum_{n=-\infty}^{\infty} u[n] z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Differentiating w.r.t. z on both sides we have,

$$\sum_{n=-\infty}^{\infty} n u[n] z^{-n-1} = \frac{1}{(z - 1)^2}$$

$$\implies \sum_{n=-\infty}^{\infty} n u[n] z^{-n} = \frac{z}{(z - 1)^2}$$

Repeating the differentiation k-1 more times we get,

$$Y(z) = n^k u[n] z^{-n} = \frac{f(z)}{(z - 1)^{k+1}}$$

for some polynomial $f(z)$.

Therefore, $z = 1$ is $(k+1)^{\text{th}}$ order pole. Therefore the system has $k+1$ modes.

Taking Z-transform of $u[n]$ we get,

$$u[n] \to \frac{1}{1 - z^{-1}}$$

Similarly,

$$n\,u[n] \to \frac{z}{(z-1)^2}$$

and

$$n^2 u[n] \to \frac{z(z+1)}{(z-1)^3}$$

Generalizing this for k^{th} power of n, we get,

$$n^k u[n] \to \frac{1}{(z-1)^{k+1}} \sum_{n=0}^{k-1} \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle z^{n+1}$$

where $\left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle$ for $n = 0$ to $n = k-1$ are Eulerian numbers.

So,

$$\frac{Y(z)}{X(z)} = \frac{1}{(1 - z^{-1})^{k+1}} \sum_{n=0}^{k-1} \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle z^{n-k}$$

$$\implies (1 - z^{-1})^{k+1} Y(z) = \sum_{n=0}^{k-1} \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle z^{n-k} X(z)$$

$$\implies \binom{k+1}{0} Y(z) - \binom{k+1}{1} z^{-1} Y(z) + \binom{k+1}{2} z^{-2} Y(z) - \binom{k+1}{3} z^{-3} Y(z) + \cdots + (-1)^{k+1} \binom{k+1}{k+1} z^{-(k+1)} Y(z)$$

$$= \left\langle \begin{matrix} k \\ 0 \end{matrix} \right\rangle z^{-k} X(z) + \left\langle \begin{matrix} k \\ 1 \end{matrix} \right\rangle z^{-(k-1)} X(z) + \left\langle \begin{matrix} k \\ 2 \end{matrix} \right\rangle z^{-(k-2)} X(z) + \cdots + \left\langle \begin{matrix} k \\ k-1 \end{matrix} \right\rangle z^{-1} X(z)$$

Thus we get difference equation of the system as,

$$\binom{k+1}{0} y[n] - \binom{k+1}{1} y[n-1] + \binom{k+1}{2} y[n-2] - \binom{k+1}{3} y[n-3] + \cdots + (-1)^{k+1} y[n-(k+1)]$$

$$= \left\langle \begin{matrix} k \\ 0 \end{matrix} \right\rangle x[n-k] + \left\langle \begin{matrix} k \\ 1 \end{matrix} \right\rangle x[n-(k-1)] + \left\langle \begin{matrix} k \\ 2 \end{matrix} \right\rangle x[n-(k-2)] + \cdots + \left\langle \begin{matrix} k \\ k-1 \end{matrix} \right\rangle x[n-1]$$

$$\implies \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} y[n-i] = \sum_{j=0}^{k-1} \left\langle \begin{matrix} k \\ j \end{matrix} \right\rangle x[n-(k-j)]$$

PROBLEM 3:
Three coins C_1, C_2 and C_3 each show heads with probability 3/5 and tails otherwise. C_1 counts 10 points for a head

and 2 for a tail, C_2 counts 4 points for both head and tail and C_3 counts 3 points for a head and 20 for a tail. You and your opponent each choose a coin; you cannot choose the same coin as your opponent. Each of you tosses your coin and the person with the larger score wins 10 dollars. Would you prefer to be the first to pick a coin or the second? Explain. (5 pts.)

Solution:

It may appear intuitively that the person who goes first has more chances of winning. But this is not true. Let us discuss different cases.

Case 1: Your opponent chooses C_1. In this case, you should choose C_3.

Reason: Probability that C_3 beats C_1 = P(HT or TH or TT) = P(HT) + P(TH) + P(TT) = $\frac{6+6+4}{25} = \frac{16}{25}$.

Case 2: Your opponent chooses C_2. In this case, you should choose C_1.

Reason: Probability that C_1 beats C_2 = P(HH or TH) = P(HH) + P(TH) = $\frac{9+6}{25} = \frac{3}{5}$.

Case 3: Your opponent chooses C_3. In this case, you should choose C_2.

Reason: Probability that C_2 beats C_3 = P(HH or HT) = P(HH) + P(HT) = $\frac{9+6}{25} = \frac{3}{5}$.

No matter which coin your opponent choses, you can choose a coin which has more probability of winning against your opponent's coin. Thus, you would prefer to let your opponent have the first go.

PROBLEM 4:
Three companies A, B, C manufacture light bulbs and have a market share in the ratio $0.35 : 0.35 : 0.3$. Probability of each of them producing a defective bulb is respectively $0.01, 0.02$ and 0.05. A randomly chosen bulb is found defective. What is the probability that it was manufactured by company B? (4 pts.)

Solution:

Let P(A), P(B) and P(C) denote the probabilities that a bulb has been manufactured by company A, B and C respectively. Let P(D) denote the probability that a certain bulb is defective.

We know, P(D|A) = 0.35, P(D|B) = 0.35 and P(D|C) = 0.3. We need to evaluate P(B|D).

Applying Bayes' theorem

$$\begin{aligned} P(B|D) &= \frac{P(D|B) \times P(B)}{P(D|A) \times P(A) + P(D|B) \times P(B) + P(D|C) \times P(C)} \\ &= \frac{0.02 \times 0.35}{0.01 \times 0.35 + 0.02 \times 0.35 + 0.05 \times 0.3} \\ &= \frac{14}{51} \end{aligned}$$

Thus if a randomly chosen bulb is found defective, probability that it was manufactured by company B is $\frac{14}{51}$.

PROBLEM 5:
(a) If A and B are independent events. Show that A^c and B^c are independent. (3 pts.)

(b) If A and B are two events in Ω, show that (i)$(A \cup B)^c = A^c \cap B^c$ (ii)$(A \cap B)^c = A^c \cup B^c$ (4 pts.)

Solution:
(a)

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= P(A^c) - P(B)(1 - P(A)) \\ &= P(A^c) - P(B)P(A^c) \\ &= P(A^c)P(B^c) \end{aligned}$$

(b) We show $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$

Let $x \in (A \cup B)^c \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B \Rightarrow x \in A^c$ and $x \in B^c \Rightarrow x \in A^c \cap B^c$

$(A \cup B)^c \subseteq A^c \cap B^c$

Let $x \in A^c \cap B^c \Rightarrow x \in A^c$ and $x \in B^c \Rightarrow x \notin A$ and $x \notin B \Rightarrow x \notin A \cup B \Rightarrow x \in (A \cup B)^c$

$A^c \cap B^c \subseteq (A \cup B)^c$

$\therefore (A \cup B)^c = A^c \cap B^c$

We show $(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$

Let $x \in (A \cap B)^c \Rightarrow x \notin (A \cap B) \Rightarrow x \notin A$ or $x \notin B \Rightarrow x \in A^c$ or $x \in B^c \Rightarrow x \in A^c \cup B^c$

$(A \cap B)^c \subseteq A^c \cup B^c$

Let $x \in A^c \cup B^c \Rightarrow x \in A^c$ or $x \in B^c \Rightarrow x \notin A$ or $x \notin B \Rightarrow x \notin A \cap B \Rightarrow x \in (A \cap B)^c$

$A^c \cup B^c \subseteq (A \cap B)^c$

$\therefore (A \cap B)^c = A^c \cup B^c$

PROBLEM 6:
If X is a continous random variable with pdf given by $f(t) = \frac{1}{\pi(1+t^2)}$, for $-\infty < t < \infty$. What can you say about the mean and variance of X? (6 pts.)

Solution:
This is an example of a random variable having Cauchy distribution.

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} t \frac{1}{\pi(1+t^2)} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2t \frac{1}{(1+t^2)} dt \\ &= \frac{1}{2\pi} \ln(1+t^2)|_{-\infty}^{\infty}\end{aligned}$$

This is undefined.
Similary, variance $\sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
Compute $\mathbb{E}[X^2]$

$$\begin{aligned}\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} t^2 \frac{1}{\pi(1+t^2)} dt \\ &= \int_{-\infty}^{\infty} \frac{1+t^2-1}{\pi(1+t^2)} dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(1 - \frac{1}{(1+t^2)}\right) dt \\ &= \frac{1}{\pi} \left(t|_{-\infty}^{\infty} - \tan^{-1} t|_{-\infty}^{\infty} \right)\end{aligned}$$

This is also undefined. Therefore, both the mean and variance are undefined.

PROBLEM 7:
Consider 2D points lying on a circular disk of radius R centered at origin.

$$S \coloneqq \left\{(x, y) : x^2 + y^2 \leq R^2\right\}$$

All points $(x, y) \in S$ are uniformly distributed on the disk. Obtain the marginal distributions $f_X(x)$ and $f_Y(y)$. Are they statistically independent? Are they correlated? (8 pts.)

Solution:
Join pdf of random variables X and Y uniformly distributed in S can be written as

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2} \text{ for } (x, y) \in S \text{ i.e., } x^2 + y^2 \leq R^2 \\ 0 \text{ otherwise} \end{cases}$$

The marginal probability density for random variable X can be evaluated as

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\
&= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2} dy \qquad \bigg(\text{probability of points outside } S \text{ is zero}\bigg) \\
&= \frac{2}{\pi R^2}\sqrt{R^2-x^2}
\end{aligned}
$$

Therefore, including points of measure 0, we have

$$
f_X(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-x^2} \text{ for } -R \le x \le R \\ 0 \text{ otherwise} \end{cases}
$$

Similary, the marginal probability density for random variable Y can be evaluated as

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\
&= \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{1}{\pi R^2} dx \qquad \bigg(\text{probability of points outside } S \text{ is zero}\bigg) \\
&= \frac{2}{\pi R^2}\sqrt{R^2-y^2}
\end{aligned}
$$

Therefore, including points of measure 0, we have

$$
f_X(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-y^2} \text{ for } -R \le y \le R \\ 0 \text{ otherwise} \end{cases}
$$

Since $f_{XY}(x,y) \neq f_X(x)f_Y(y)$, we infer that X and Y are not independent.
To check for correlation, we evaluate $\mathbb{E}[XY]$

$$
\begin{aligned}
\mathbb{E}[XY] &= \int_x \int_y xy f_{XY}(x,y)dxdy \\
&= \int_{-R}^{R} x\bigg(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{y}{\pi R^2} dy\bigg)dx \\
&= 0 \qquad \bigg(\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{y}{\pi R^2} dy = 0 \text{ since integral of odd function}\bigg) \\
\mathbb{E}[X] &= \int_{-R}^{R} x\bigg(\frac{2\sqrt{R^2-x^2}}{\pi R^2}\bigg)dx = 0 \qquad \bigg(\text{integral of odd function}\bigg) \\
\mathbb{E}[Y] &= \int_{-R}^{R} x\bigg(\frac{2\sqrt{R^2-y^2}}{\pi R^2}\bigg)dy = 0 \qquad \bigg(\text{integral of odd function}\bigg)
\end{aligned}
$$

Covariance $\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$. Therefore, X and Y are uncorrelated.

PROBLEM 8:
Consider two uniformly distributed $U[0,1]$ random variables X and Y. Let us form a new random variable $Z = |X-Y|$. Find and sketch (a) Probability distribution function of Z (b) Probability density function of Z. Also compute the mean and variance of Z. (10 pts.)

Solution:
(a)Probability distribution function of Z:
$F_Z(z) = P(Z \le z)$. We note that as $Z = |X-Y|$ for $z \ge 1$, $|X-Y| \le z$ with probability 1. We evaluate below the

distribution function for $z \leq 1$.

$$
\begin{aligned}
F_Z(z) &= P(Z \leq z) = P(|X-Y| \leq z) \\
&= P(-z \leq X - Y \leq z) \\
&= P\big(\{-z \leq X - Y\} \cap \{X - Y \leq z\}\big) \\
&= 1 - P\big(\{-z \leq X - Y\}^c \cup \{X - Y \leq z\}^c\big) \\
&= 1 - P\big(\{-z \leq X - Y\}^c\big) - P\big(\{X - Y \leq z\}^c\big) + P\big(\{-z \leq X - Y\}^c \cap \{X - Y \leq z\}^c\big) \\
&= 1 - P\big(\{X - Y < -z\}\big) - P\big(\{X - Y > z\}\big) + P\big(\{X - Y < -z\} \cap \{X - Y > z\}\big) \\
&= 1 - P\big(\{X - Y < -z\}\big) - P\big(\{X - Y > z\}\big)
\end{aligned}
$$

Let us evaluate $P\big(\{X - Y < -z\}\big)$

$$
\begin{aligned}
P\big(\{X - Y < -z\}\big) &= \int_{y=z}^{1} \int_{x=0}^{y-z} f_{XY}(x,y) dy\, dx \\
&= \int_{y=z}^{1} \int_{x=0}^{y-z} 1 \cdot dx\, dy \\
&= \int_{x=z}^{1} (y-z)\, dy = \frac{1}{2} - z - z^2
\end{aligned}
$$

Let us evaluate $P\big(\{X - Y > z\}\big)$

$$
\begin{aligned}
P\big(\{X - Y > z\}\big) &= \int_{x=z}^{1} \int_{y=0}^{x-z} f_{XY}(x,y) dy\, dx \\
&= \int_{x=z}^{1} \int_{y=0}^{x-z} 1 \cdot dy\, dx \\
&= \int_{x=z}^{1} (x-z)\, dx = \frac{1}{2} - z - z^2
\end{aligned}
$$

Therefore, $F_Z(z) = 1 - P(X - Y > z) - P(X - Y < -z) = 2z - z^2$ for $0 \leq z \leq 1$.

$$
F_Z(z) = \begin{cases} 0 \text{ for } z < 0 \\ 2z - z^2 \text{ for } 0 \leq z \leq 1 \\ 1 \text{ for } z \geq 1 \end{cases} \tag{1}
$$

(b)Probability density function of Z:

$$
f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 2 - 2z \text{ for } 0 \leq z \leq 1 \\ 0 \text{ otherwise} \end{cases}
$$

Computation of mean and variance:

$$
\begin{aligned}
\mathbb{E}[Z] &= \int_z z f_Z(z) dz \\
&= \int_{z=0}^{1} z(2-2z) dz = \frac{1}{3} \\
\mathbb{E}[Z^2] &= \int_z z^2 F_Z(z) dz \\
&= \int_{z=0}^{1} z^2(2-2z) dz = \frac{1}{6} \\
\text{variance} &= \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 \\
&= \frac{1}{6} - \frac{1}{9} = \frac{1}{18}
\end{aligned}
$$