Indian Institute of Science

E9-207: Basics of Signal Processing

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Mid Term Exam#2 Solutions, Spring 2018

Name and SR.No:

Instructions:

- You are allowed only 4 pages of written notes and a calculator for this exam. No wireless allowed.
- The time duration is 3 hrs.
- There are six main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- Do not panic, do not cheat.
- Good luck!

Question No.	Points scored
1	
2	
3	
4	
5	
6	
Bonus	
Total points	

Solutions prepared by: Amrutha, Priya, Prayag, Shounak, Chaitanya

PROBLEM 1: Simplify the following multirate systems shown in Figure 1 as best as you can. Obtain the simplified frequency response. Show all your steps carefully. (15 pts.)

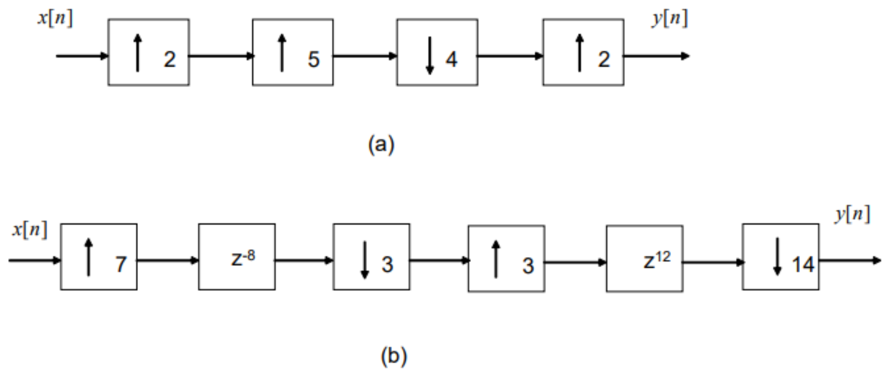


FIGURE 1. Multirate systems.

SOLUTION:

We use the identities in Figure 2 to simplify the given transformations.

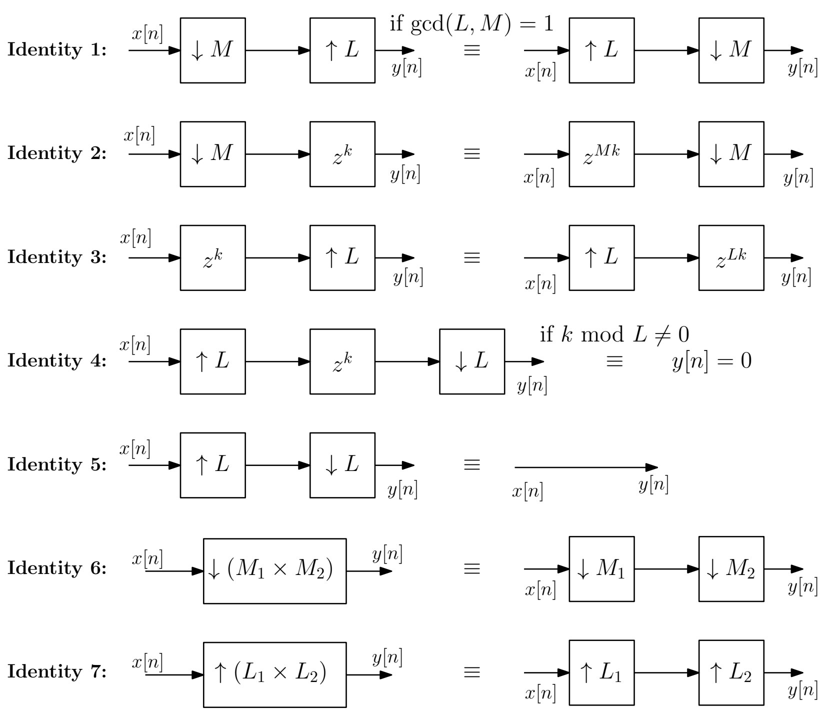
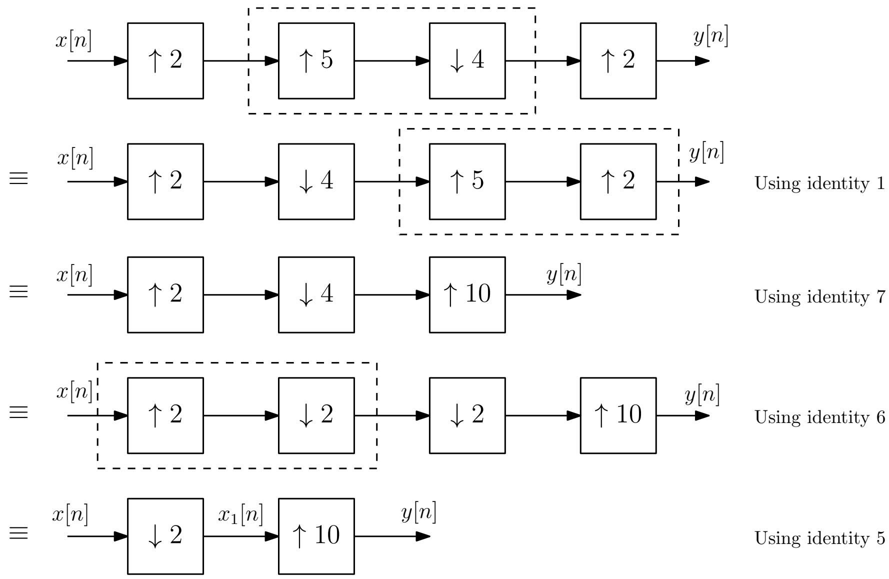


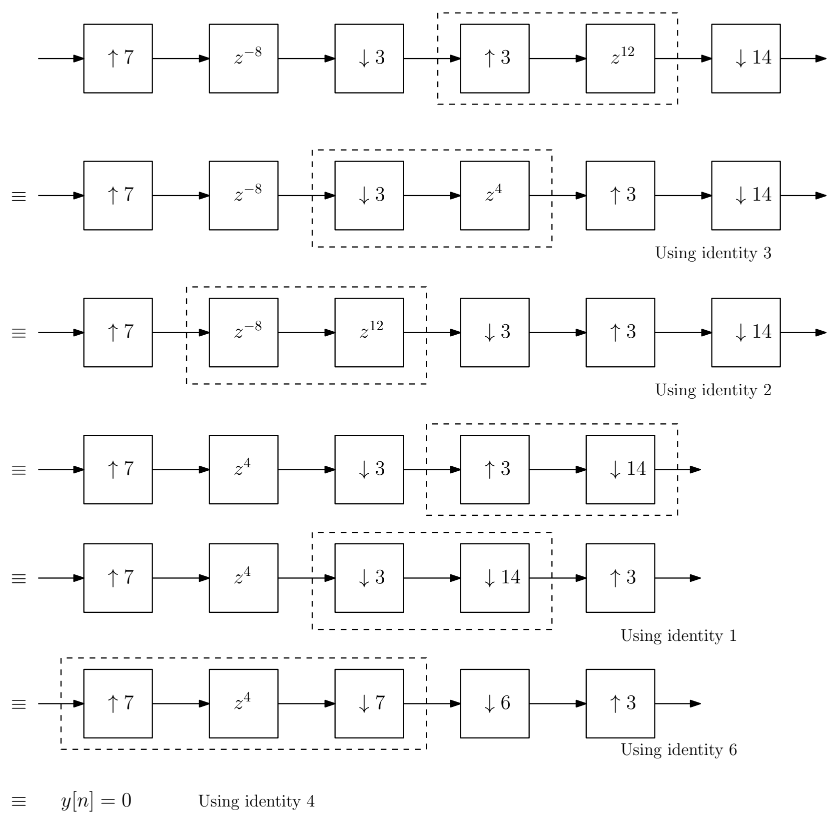
Figure 2: Identities related to decimation, upsampling and delay operations.

(Part a)



$X_1(z) = \frac{1}{2}\left(X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\right)$
Therefore, $Y(z) = X_1\left(z^{10}\right) = \frac{1}{2}\left(X\left(z^5\right) + X\left(-z^5\right)\right)$.

(Part b)



Therefore, $Y(z) = 0$.

PROBLEM 2: Prove that decimation by M followed by expansion by L can be interchanged if L and M are relatively prime. You must prove this result in the frequency domain representation. (10 pts.)

SOLUTION:

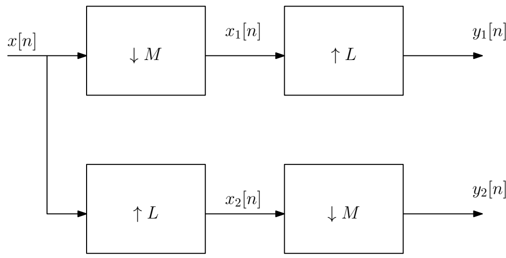


Figure 3: Comparing the outputs by changing the order of decimator and upsampler.

From Figure 3,

$$\begin{aligned} X_1(z) &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right) \\ \implies Y_1(z) &= X_1\left(z^L\right) \\ &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(z^{\frac{L}{M}} e^{j\frac{2\pi i}{M}}\right). \end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned} X_2(z) &= X\left(z^L\right) \\ Y_2(z) &= \frac{1}{M}\sum_{i=0}^{M-1} X_2\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right) \\ &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right)^L\right) \\ &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(z^{\frac{L}{M}} e^{j\frac{2\pi i L}{M}}\right). \end{aligned} \tag{2}$$

To prove that $Y_1(z) = Y_2(z)\,\forall X(z)$, it is necessary and sufficient to satisfy the following condition:

$$\left\{X\left(z^{\frac{L}{M}} e^{j\frac{2\pi i L}{M}}\right) \mid i = 0, 1, \cdots, M-1\right\} = \left\{X\left(z^{\frac{L}{M}} e^{j\frac{2\pi i}{M}}\right) \mid i = 0, 1, \cdots, M-1\right\} \forall X(z) \tag{3}$$

$$i.e., \left\{e^{j\frac{2\pi i L}{M}} \mid i = 0, 1, \cdots, M-1\right\} = \left\{e^{j\frac{2\pi i}{M}} \mid i = 0, 1, \cdots, M-1\right\}. \tag{4}$$

Since $e^{j2\pi k} = 1 \forall k \in \mathbf{Z}$, we have $e^{j\frac{2\pi i L}{M}} = e^{j\frac{2\pi (iL \mod M)}{M}}$. Hence, the equivalent condition is

$$\{(iL) \mod M \mid i = 0, 1, \cdots, M-1\} = \{0, 1, \cdots, M-1\}. \tag{5}$$

Let $0 \le i_1 \le M-1$ and $0 \le i_2 \le M-1$ such that $i_1 \ne i_2$. Without loss of generality, consider $i_1 < i_2$. Using the following identity on modulo operation

$$(a-b) \mod M = (a \mod M - b \mod M) \mod M,$$

we have,

$$((i_1 L) \mod M - (i_2 L) \mod M) \mod M = ((i_1 - i_2) L) \mod M. \tag{6}$$

Case L and M are relatively prime:
Since $0 < i_1 - i_2 < M$, and $\gcd(L, M) = 1$, $((i_1 - i_2) L) \mod M \neq 0$. Therefore from (6),

$$((i_1 L) \mod M - (i_2 L) \mod M) \mod M \neq 0,$$

$$\implies (i_1 L) \mod M \neq (i_2 L) \mod M.$$

We have proved that $i_1 \neq i_2 \implies (i_1 L) \mod M \neq (i_2 L) \mod M \, \forall i_1, i_2 \in \{0, 1, 2 \cdots, M-1\}$. Therefore, when $\gcd(L, M) = 1$, equation (5) holds true.
Case M divides L:
Let $L = P \times M$, $P > 1$. Therefore, it is possible to chose $i_1 = i_2 + M$. Under this condition,

$$((i_1 - i_2) L) \mod M = (ML) \mod M = 0.$$

Therefore,

$$((i_1 L) \mod M - (i_2 L) \mod M) \mod M = 0$$
$$\implies (i_1 L) \mod M = (i_2 L) \mod M.$$

We have shown that for some choice of $i_1 \neq i_2$, $(i_1 L) \mod M = (i_2 L) \mod M$. Hence, the values $\{(iL) \mod M\}_{i=0}^{M-1}$ are not distinct. Therefore, when M divides L, equation (5) does not hold true.
Case $\gcd(M, L) = G > 1$:
Let $M = G \times P_M$ and $L = G \times P_L$. We can chose $i_1 = i_2 + G$. Under this condition, $e^{j2\pi \frac{iL}{M}} = e^{j2\pi \frac{iP_L}{P_M}}$. Therefore, $\left\{ e^{j2\pi \frac{iP_L}{P_M}} \mid i = 0, 1, \cdots, M-1 \right\}$ has P_M distinct values. Therefore, equation (5) does not hold true under this condition.
Hence, the equation (5) holds true iff L and M are relatively prime. This proves that M fold decimator and L fold upsampler blocks can be interchanged iff L and M are relatively prime.

PROBLEM 3: This problem has two parts

- A student was performing measurements on an oscillator. Over what period must the signal be averaged so that he can claim that the device was producing frequencies accurately up to 0.25 KHz? (5 pts.)
- Suppose we are filtering a natural image using (a) 2D FFT (b) 2D Haar wavelet. Sketch the frequency resolution in the 2D $\omega_1 - \omega_2$ frequency plane for both the cases. You can assume a N-level dyadic decomposition. (15pts.)

SOLUTION:

a) By the time frequency uncertainty principle, we know that $\sigma_\omega^2 \sigma_t^2 \geq \frac{1}{4}$. Thus, if a student should claim that the device is producing frequencies accurately upto 0.25KHz, then,

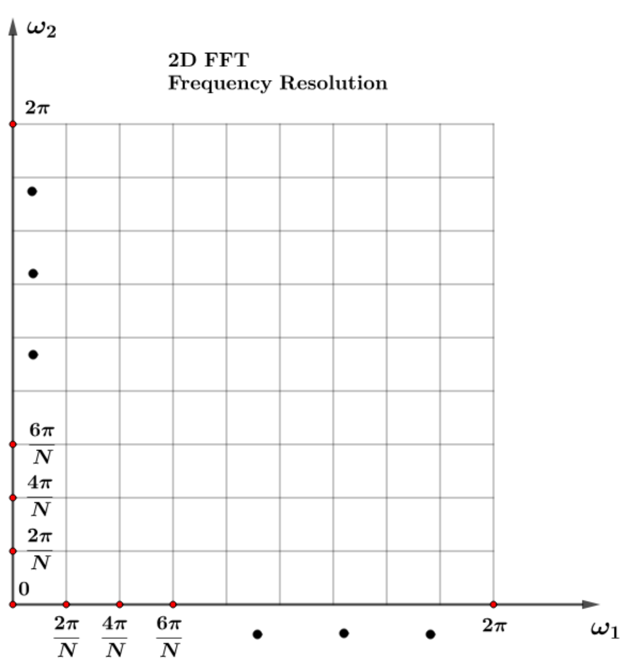
$$\begin{aligned}
\sigma_\omega \sigma_t &\geq \frac{1}{2} \Rightarrow \sigma_f \sigma_t \geq \frac{1}{4\pi} \Rightarrow 250 \times \sigma_t \geq \frac{1}{4\pi} \quad (\text{As } \sigma_f \leq 0.25\text{KHz}) \\
\sigma_t &\geq \frac{1}{4\pi \times 250} = 318.31\mu\text{s}
\end{aligned}$$

Thus, the signal should be averaged over a period of atleast 318.31 μs to claim that the device produces frequencies accurately upto 0.25 KHz.
The basic idea of this problem was to test the time-frequency uncertainty principle. If any variation of the uncertainty principle has been used, then marks have been awarded.

b) Consider the 2D image to be given by $\{x[m,n]\}_{m,n}$, then N-point 2D DFT is given as follows:

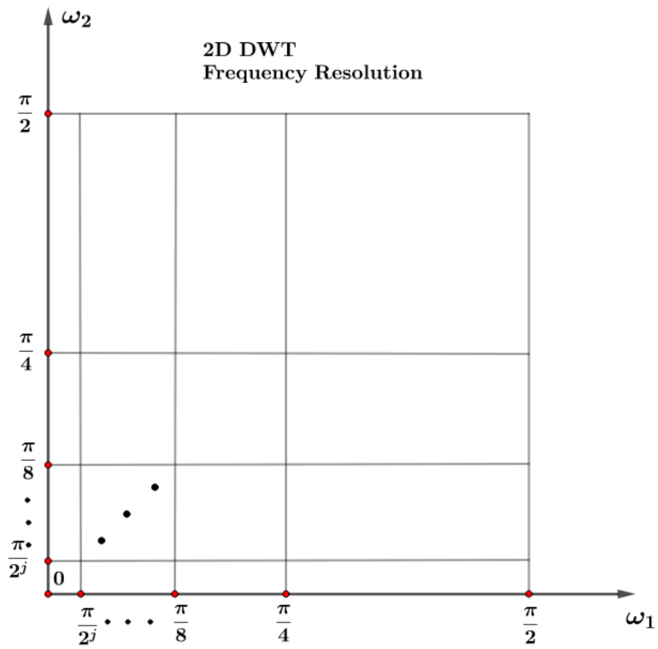
$$X[k,l] = \frac{1}{\sqrt{NM}} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x[m,n] \mathrm{e}^{-j2\pi\left[\frac{mk+nl}{N}\right]}$$

Note that the FFT representation corresponds to the coefficients of the signal expansion in terms of the exponential basis signals, namely, $\mathrm{e}^{-j2\pi\left[\frac{mk+nl}{N}\right]}$. These basis signals have frequency increasing uniformly. Thus, the frequency resolution in the 2D $\omega_1 - \omega_2$ frequency plane is uniform as shown in the figure below.



Similarly, the DWT representation corresponds to the coefficients of the signal expansion in terms of the Haar wavelet basis signals along the two axes of the image. In 1D, consider the signal to have $M = 2^N$ samples.

The basis signals are given by $\{\psi(t), \{\psi(2^{-1}t-k)\}_k, \ldots, \{\psi(2^{N-1}t-k)\}_k\}$. The frequency resolution of the basis signals are decreasing dyadically. Note that for a discrete signal, the sampling frequency corresponds to 2π. As image and music signals are low frequency signals, we consider the relative frequencies only from 0 to $\frac{\pi}{2}$. Thus, the frequency resolution in the frequency plane is dyadically spaced from $\frac{\pi}{2^N}$ to $\frac{\pi}{2}$. Note that for the j^{th} level, the frequency resolution is $\frac{\pi}{2^j}$. Similarly, for 2D too the frequency resolution in the 2D $\omega_1 - \omega_2$ frequency plane is dyadically spaced as shown in the figure below.



PROBLEM 4: Consider a 3-channel filterbank with analysis filters $H_0(z), H_1(z)$ and $H_2(z)$ across first three branches respectively. Consider decimation rates over the three branches to be (a) $(2, 3, 6)$ (b) $(2, 4, 4)$. Derive the conditions for alias free reconstruction from first principles. If the synthesis filters are $F_0(z), F_1(z)$ and $F_2(z)$, obtain the alias free distortion function. (25 pts.)

SOLUTION:

(a)

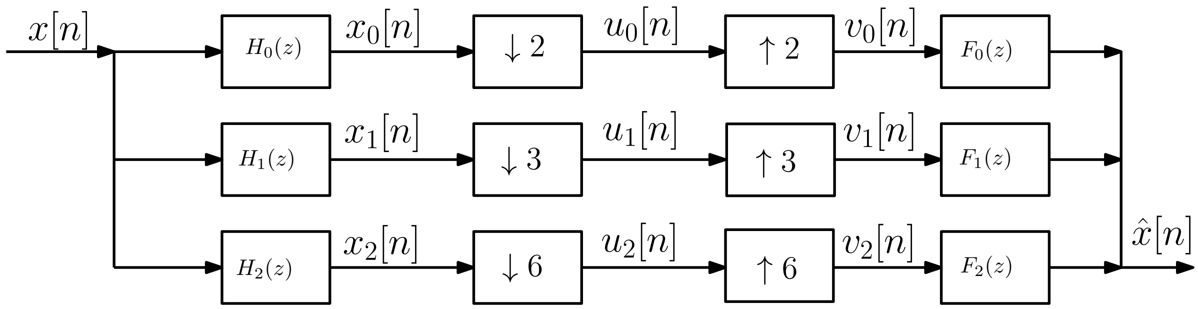


Figure 4: 3-channel filterbank with decimation rates (2,3,6)

Analysis filters - $H_0(z),\ H_1(z),\ H_2(z)$
Synthesis filters - $F_0(z),\ F_1(z),\ F_2(z)$
From Figure 4,

$$\begin{aligned} X_0(z) &= H_0(z)X(z), \\ X_1(z) &= H_1(z)X(z), \\ X_2(z) &= H_2(z)X(z). \end{aligned} \tag{7}$$

$u_k[n]$ is obtained by downsampling $x_k[n]$ where $k = 0, 1, 2$.
For downsampling by M,

$$U_M(z) = \frac{1}{M}\sum_{k=0}^{M-1} X_0\left(z^{\frac{1}{M}}\omega_M^k\right), \tag{8}$$

where $\omega_M^k = e^{\frac{j2\pi k}{M}}$. After downsampling, we get

$$\begin{aligned} U_0(z) &= \frac{1}{2}\sum_{k=0}^{1} X_0\left(z^{\frac{1}{2}}\omega_2^k\right), \\ U_1(z) &= \frac{1}{3}\sum_{k=0}^{2} X_1\left(z^{\frac{1}{3}}\omega_3^k\right), \\ U_2(z) &= \frac{1}{6}\sum_{k=0}^{5} X_2\left(z^{\frac{1}{6}}\omega_6^k\right). \end{aligned} \tag{9}$$

$v_k[n]$ is obtained by upsampling $u_k[n]$ where $k = 0, 1, 2$.
For upsampling by L,

$$V_k(z) = U_k(z^L) \tag{10}$$

After upsampling, we get

$$
\begin{aligned}
V_0(z) = U_0(z^2) &= \frac{1}{2}\sum_{k=0}^{1} X_0\left(z\omega_2^k\right), \\
V_1(z) = U_1(z^3) &= \frac{1}{3}\sum_{k=0}^{2} X_1\left(z\omega_3^k\right), \\
V_2(z) = U_2(z^6) &= \frac{1}{6}\sum_{k=0}^{5} X_2\left(z\omega_6^k\right).
\end{aligned} \tag{11}
$$

At the output, we have

$$
\begin{aligned}
\hat{X}(z) &= \sum_{l=0}^{2} F_l(z)V_l(z) \\
&= F_0(z)V_0(z) + F_1(z)V_1(z) + F_2(z)V_2(z) \\
&= \left(F_0(z) \times \frac{1}{2}\sum_{k=0}^{1} X_0\left(z\omega_2^k\right)\right) + \left(F_1(z) \times \frac{1}{3}\sum_{k=0}^{2} X_1\left(z\omega_3^k\right)\right) \\
&\quad + \left(F_2(z) \times \frac{1}{6}\sum_{k=0}^{5} X_2\left(z\omega_6^k\right)\right)
\end{aligned} \tag{12}
$$

Using (7) in (12) we get

$$
\begin{aligned}
\hat{X}(z) = &\left(F_0(z) \times \frac{1}{2}\sum_{k=0}^{1} H_0\left(z\omega_2^k\right) X\left(z\omega_2^k\right)\right) + \left(F_1(z) \times \frac{1}{3}\sum_{k=0}^{2} H_1\left(z\omega_3^k\right) X\left(z\omega_3^k\right)\right) \\
&+ \left(F_2(z) \times \frac{1}{6}\sum_{k=0}^{5} H_2\left(z\omega_6^k\right) X\left(z\omega_6^k\right)\right) \\
&= F_0(z) \times \frac{1}{2}\sum_{k=0}^{1} H_0(z)X(z) + F_0(z) \times \frac{1}{2}\sum_{k=0}^{1} H_0(z)X(z\omega_2^k) \\
&+ F_1(z) \times \frac{1}{3}\sum_{k=0}^{2} H_1(z)X(z) + F_1(z) \times \frac{1}{3}\sum_{k=0}^{2} H_1(z)X(z\omega_3^k) \\
&+ F_2(z) \times \frac{1}{6}\sum_{k=0}^{5} H_2(z)X(z) + F_2(z) \times \frac{1}{6}\sum_{k=0}^{5} H_2(z)X(z\omega_6^k)
\end{aligned}
$$

In the expression of $\hat{X}(z)$ above, all the terms scaling with ω_M^k result in aliasing. Hence for zero aliasing, these terms should be zero. Alias free condition:

$$
\frac{1}{2}H_0(-z)F_0(z) + \frac{1}{3}F_1(z)\sum_{k=1}^{2} H_1(z\omega_3^k) + \frac{1}{6}F_2(z)\sum_{k=1}^{5} H_2(z\omega_6^k) = 0 \tag{13}
$$

For alias free distortion function, consider

$$
\hat{X}(z) = X(z)T(z), \tag{14}
$$

where $T(z)$ results in alias free output $\hat{X}(z)$ from $X(z)$. When condition (13) is satisfied, we get

$$
\hat{X}(z) = \frac{1}{2}H_0(z)F_0(z)X(z) + \frac{1}{3}H_1(z)F_1(z)X(z) + \frac{1}{6}H_2(z)F_2(z)X(z) \tag{15}
$$

$$
= \left[\frac{1}{2}H_0(z)F_0(z) + \frac{1}{3}H_1(z)F_1(z) + \frac{1}{6}H_2(z)F_2(z)\right] X(z) \tag{16}
$$

Comparing (14) and (16) we get Alias free distortion function:

$$T(z) = \frac{1}{2}H_0(z)F_0(z) + \frac{1}{3}H_1(z)F_1(z) + \frac{1}{6}H_2(z)F_2(z) \tag{17}$$

(b)

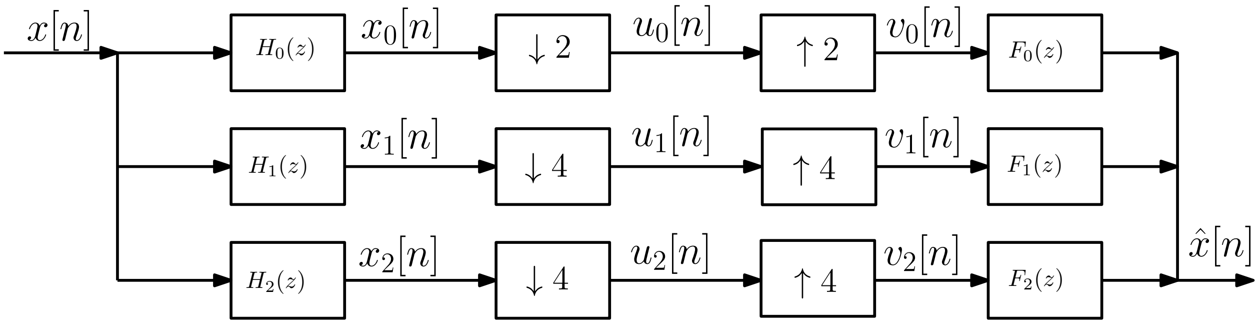


Figure 5: 3-channel filterbank with decimation rates (2,4,4)

Analysis filters - $H_0(z),\ H_1(z),\ H_2(z)$
Synthesis filters - $F_0(z),\ F_1(z),\ F_2(z)$
From Figure 5,

$$\begin{aligned} X_0(z) &= H_0(z)X(z), \\ X_1(z) &= H_1(z)X(z), \\ X_2(z) &= H_2(z)X(z). \end{aligned} \tag{18}$$

$u_k[n]$ is obtained by downsampling $x_k[n]$ where $k = 0, 1, 2$.
For downsampling by M,

$$U_M(z) = \frac{1}{M}\sum_{k=0}^{M-1} X_0\left(z^{\frac{1}{M}}\omega_M^k\right), \tag{19}$$

where $\omega_M^k = e^{\frac{j2\pi k}{M}}$. After downsampling, we get

$$\begin{aligned} U_0(z) &= \frac{1}{2}\sum_{k=0}^{1} X_0\left(z^{\frac{1}{2}}\omega_2^k\right), \\ U_1(z) &= \frac{1}{4}\sum_{k=0}^{3} X_1\left(z^{\frac{1}{4}}\omega_4^k\right), \\ U_2(z) &= \frac{1}{4}\sum_{k=0}^{3} X_2\left(z^{\frac{1}{4}}\omega_4^k\right). \end{aligned} \tag{20}$$

$v_k[n]$ is obtained by upsampling $u_k[n]$ where $k = 0, 1, 2$.
For upsampling by L,

$$V_k(z) = U_k(z^L) \tag{21}$$

After upsampling, we get

$$V_0(z) = U_0(z^2) = \frac{1}{2}\sum_{k=0}^{1} X_0\left(z\omega_2^k\right),$$
$$V_1(z) = U_1(z^4) = \frac{1}{4}\sum_{k=0}^{3} X_1\left(z\omega_4^k\right),$$
$$V_2(z) = U_2(z^4) = \frac{1}{4}\sum_{k=0}^{3} X_2\left(z\omega_4^k\right). \tag{22}$$

At the output, we have

$$\begin{aligned}\hat{X}(z) &= \sum_{l=0}^{2} F_l(z)V_l(z)\\ &= F_0(z)V_0(z) + F_1(z)V_1(z) + F_2(z)V_2(z)\\ &= \left(F_0(z) \times \frac{1}{2}\sum_{k=0}^{1} X_0\left(z\omega_2^k\right)\right) + \left(F_1(z) \times \frac{1}{4}\sum_{k=0}^{3} X_1\left(z\omega_4^k\right)\right)\\ &\quad + \left(F_2(z) \times \frac{1}{4}\sum_{k=0}^{3} X_2\left(z\omega_4^k\right)\right)\end{aligned} \tag{23}$$

Using (18) in (23) we get

$$\begin{aligned}\hat{X}(z) = &\left(F_0(z) \times \frac{1}{2}\sum_{k=0}^{1} H_0\left(z\omega_2^k\right)X\left(z\omega_2^k\right)\right) + \left(F_1(z) \times \frac{1}{4}\sum_{k=0}^{3} H_1\left(z\omega_4^k\right)X\left(z\omega_4^k\right)\right)\\ &+ \left(F_2(z) \times \frac{1}{4}\sum_{k=0}^{3} H_2\left(z\omega_4^k\right)X\left(z\omega_4^k\right)\right)\end{aligned} \tag{24}$$

We have $\omega_M^0 = 1$ and the terms that result in aliasing are of the form $X(z\omega_M^k), k \neq 0$. From (24), the alias free condition is

$$\frac{1}{2}H_0(-z)F_0(z) + \frac{1}{4}\sum_{k=1}^{3} H_1(z\omega_3^k)F_1(z) + \frac{1}{4}\sum_{k=1}^{3} H_2(z\omega_4^k)F_2(z) = 0 \tag{25}$$

To obtain the alias free distortion function, consider

$$\hat{X}(z) = X(z)T(z), \tag{26}$$

where $T(z)$ results in alias free output $\hat{X}(z)$ from $X(z)$. When condition (25) is satisfied, we get

$$\hat{X}(z) = \left[\frac{1}{2}F_0(z)H_0(z) + \frac{1}{4}F_1(z)H_1(z) + \frac{1}{4}F_2(z)H_2(z)\right]X(z) \tag{27}$$

Therefore, the alias free distortion function is given by,

$$T(z) = \frac{1}{2}F_0(z)H_0(z) + \frac{1}{4}F_1(z)H_1(z) + \frac{1}{4}F_2(z)H_2(z) \tag{28}$$

PROBLEM 5: Let $H_0(z) = 1 + 2z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4}$. The analysis filters are quadrature mirror symmetric. Draw an implementation for the pair $\left[\begin{array}{cc} H_0(z) & H_1(z) \end{array}\right]$ in the form of a uniform DFT analysis bank, explicitly showing the polyphase components, the 2×2 IDFT box and relevant details. (20 pts.)

SOLUTION:

We have,

$$H_0(z) = 1 + 2z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4}. \tag{29}$$

Let another analysis filter be $H_1(z)$. It is said in the question that the analysis filters are quadrature mirror symmetric, hence,

$$H_1(z) = H_0(-z) \tag{30}$$

$H_0(z)$ can be written in polyphase form, i.e.,

$$H_0(z) = E_{00}(z^2) + z^{-1}E_{01}(z^2) \tag{31}$$

Simplifying $H_0(z)$, we have,

$$H_0(z) = 1 + 4z^{-2} + z^{-4} + z^{-1}(2 + 2z^{-2}) \tag{32}$$

From equation (31) and (32) we have,

$$E_{00}(z^2) = 1 + 4z^{-2} + z^{-4} \tag{33}$$

$$E_{01}(z^2) = 2(1 + z^{-2}) \tag{34}$$

From (30) we have,

$$\begin{aligned} H_1(z) &= 1 - 2z^{-1} + 4z^{-2} - 2z^{-3} + z^{-4} \\ &= (1 + 4z^{-2} + z^{-4}) + z^{-1}(-2 - 2z^{-2}) \end{aligned} \tag{35}$$

Similarly, the polyphase components of $H_1(z)$ are given by,

$$E_{10}(z^2) = 1 + 4z^{-2} + z^{-4} \tag{36}$$

$$E_{11}(z^2) = -2 - 2z^{-2} \tag{37}$$

Comparing equations (33), (34) with equations (36), (37) respectively we obtain,

$$E_{00}(z^2) = E_{10}(z^2) \tag{38}$$

$$E_{01}(z^2) = -E_{11}(z^2) \tag{39}$$

For simplicity we consider

$$E_{00}(z^2) = E_{10}(z^2) = E_0(z^2)$$

$$E_{01}(z^2) = -E_{11}(z^2) = E_1(z^2)$$

where, $E_0(z^2) = 1 + 4z^{-2} + z^{-4}$ and $E_1(z^2) = 2(1 + z^{-2})$. Writing up in matrix form, we have

$$\left[\begin{array}{c} H_0(z) \\ H_1(z) \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right] \left[\begin{array}{c} E_0(z^2) \\ z^{-1}E_1(z^2) \end{array}\right] \tag{40}$$

Now, the two point IDFT matrix is given by,

$$\begin{bmatrix} e^{\frac{-j2\pi(0)}{2}} & e^{\frac{-j2\pi(0)}{2}} \\ e^{\frac{-j2\pi(0)}{2}} & e^{\frac{-j2\pi(1)}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = W_N^* \tag{41}$$

Therefore, we have,

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} W_N^* \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1}E_1(z^2) \end{bmatrix} \tag{42}$$

Above can be represented as shown in the Figure 6.

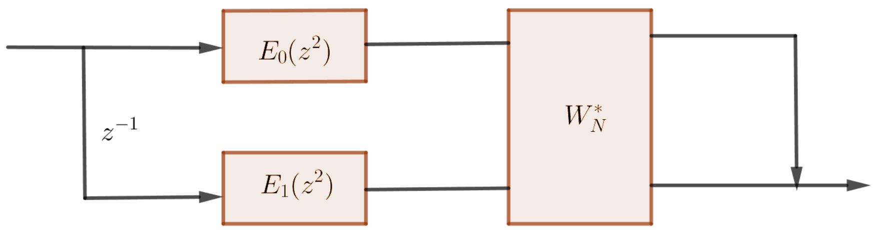


Figure 6: Implementation of the pair $\begin{bmatrix} H_0(z) & H_1(z) \end{bmatrix}$ where, $E_0(z^2) = 1 + 4z^{-2} + z^{-4}$ and $E_1(z^2) = 2(1 + z^{-2})$

PROBLEM 6: Expand the signal $s(t) = t^3$ in the interval $[0, 1]$ using Haar wavelets up to a resolution of 0.25. (10 pts.)

SOLUTION:
We have the signal $s(t) = t^3$ over the interval $[0, 1]$. We need to represent this signal in terms of the Haar wavelet basis upto a resolution of 0.25. We know that the scaling function of the Haar basis is given as follows:

$$\phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{else} \end{cases}$$

The Haar wavelet is given by:

$$\psi(t) = \begin{cases} 1 & 0 \leq t < 0.5 \\ -1 & 0.5 \leq t < 1 \\ 0 & \text{else} \end{cases}$$

The Haar wavelet basis comprises of the scaling function and the scaled and shifted versions of the Haar wavelet namely,

$$B = \left\{ \phi(t), \psi_{n,k}(t)_{n,k} \right\} \quad \text{where } n, k \in \mathbb{Z}$$

where $\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k)$. These form an orthonormal basis. Thus, we can represent the signal as follows:

$$s(t) = a\phi(t) + \sum_{n,k} b_{n,k}\psi_{n,k}(t), \text{ where } a =< \phi(t), s(t) >, b_{n,k} =< \psi_{n,k}(t), s(t) >$$

To obtain the representation for the signal $s(t)$ upto a resolution of 0.25, we need to only consider the projection of the signal onto the space spanned by $\phi(t), \psi_{0,0}(t), \psi_{1,0}(t)$ and $\psi_{1,1}(t)$ as the other signals have resolution greater than 0.25. We compute the coefficients $a, b_{0,0}, b_{1,0}$ and $b_{1,1}$ as follows:

$$\begin{aligned}
a &= \int_{-\infty}^{\infty} \phi(t)s(t)\mathrm{d}t = \int_0^1 t^3 \mathrm{d}t = \left.\frac{t^4}{4}\right|_0^1 = \frac{1}{4} \\
b_{0,0} &= \int_{-\infty}^{\infty} \psi_{0,0}(t)s(t)\mathrm{d}t = \int_0^{0.5} t^3\mathrm{d}t - \int_{0.5}^1 t^3\mathrm{d}t = \left.\frac{t^4}{4}\right|_0^{0.5} - \left.\frac{t^4}{4}\right|_{0.5}^1 = -\frac{7}{32} \\
b_{1,0} &= \int_{-\infty}^{\infty} \psi_{1,0}(t)s(t)\mathrm{d}t = \int_0^{0.25} \sqrt{2}t^3\mathrm{d}t - \int_{0.25}^{0.5} \sqrt{2}t^3\mathrm{d}t = \left.\frac{\sqrt{2}t^4}{4}\right|_0^{0.25} - \left.\frac{\sqrt{2}t^4}{4}\right|_{0.25}^{0.5} = -\frac{7}{256\sqrt{2}} \\
b_{1,1} &= \int_{-\infty}^{\infty} \psi_{1,1}(t)s(t)\mathrm{d}t = \int_{0.5}^{0.75} \sqrt{2}t^3\mathrm{d}t - \int_{0.75}^{1} \sqrt{2}t^3\mathrm{d}t = \left.\frac{\sqrt{2}t^4}{4}\right|_{0.5}^{0.75} - \left.\frac{\sqrt{2}t^4}{4}\right|_{0.75}^{1} = -\frac{55}{256\sqrt{2}}
\end{aligned}$$

Thus, the representation of the signal upto a resolution of 0.25 is given as,

$$\begin{aligned}
s_{\text{approx}}(t) &= \frac{1}{4}\phi(t) - \frac{7}{32}\psi_{0,0}(t) - \frac{7}{256\sqrt{2}}\psi_{1,0}(t) - \frac{55}{256\sqrt{2}}\psi_{1,1}(t) \\
&= \frac{1}{4}\phi(t) - \frac{7}{32}\psi(t) - \frac{7}{256}\psi(2t) - \frac{55}{256}\psi(2t-1)
\end{aligned}$$

As the question mentions Haar wavelets, the above answer is the expected answer. If the signal is expanded using the Haar basis, then partial marks is awarded. The solution using Haar basis is as follows:
As the resolution is 0.25, we consider the basis to be:

$$B = \{\phi_{2,k}(t) = 2\phi(4t - k)\}_k \quad \text{where } k \in \mathbb{Z}$$

These form an orthonormal basis. Thus, we can represent the signal as follows:

$$s(t) = \sum_k c_k \phi_{2,k}(t), \text{ where } c_k =< \phi_{2,k}(t), s(t) >$$

To obtain the representation for $s(t)$ upto a resolution of 0.25, we need to only consider the projection of the signal onto the space spanned by $\phi_{2,0}(t), \phi_{2,1}(t), \phi_{2,2}(t)$ and $\phi_{2,3}(t)$. We compute the coefficients c_0, c_1, c_2 and c_3 as follows:

$$\begin{aligned} c_k &= \int_{-\infty}^{\infty} \phi_{2,k}(t)s(t)\mathrm{d}t = \int_{k/4}^{(k+1)/4} 2t^3\mathrm{d}t = \left.\frac{t^4}{2}\right|_{k/4}^{(k+1)/4} = \frac{1}{512}((k+1)^4 - k^4) \\ \Rightarrow c_0 &= \frac{1}{512},\ c_1 = \frac{15}{512},\ c_2 = \frac{65}{512},\ c_3 = \frac{175}{512}. \end{aligned}$$

Thus, the representation of the signal upto a resolution of 0.25 is given as,

$$\begin{aligned} s'_{\text{approx}}(t) &= \frac{1}{512}\phi_{2,0}(t) + \frac{15}{512}\phi_{2,1}(t) + \frac{65}{512}\phi_{2,2}(t) + \frac{175}{512}\phi_{2,3}(t) \\ &= \frac{1}{256}\phi(4t) + \frac{15}{256}\phi(4t-1) + \frac{65}{256}\phi(4t-2) + \frac{175}{256}\phi(4t-3) \end{aligned}$$

BONUS POINTS: Examine if the function $f(t) = t^2$ is uniformly continuous over $(0, \infty)$ (10 pts.)

SOLUTION:

Given metric spaces A and B , a function $f : A \to B$ is said to be uniformly continuous if for every $\epsilon > 0$ there exist $\delta > 0$ such that for every $a, b \in A,\ |a - b| < \delta \implies |f(a) - f(b)| < \epsilon$. Consider $t_1, t_2 \in (0, \infty)$. We see that

$$\begin{aligned} |f(t_1) - f(t_2)| =& |t_1^2 - t_2^2| \\ =& |(t_1 - t_2)(t_1 + t_2)| \\ <& \delta|(t_1 + t_2)| \end{aligned} \tag{43}$$

From equation (43), we see that $\epsilon = \delta|(t_1 + t_2)|$ and this shows the dependency of δ on t_1 and ϵ. Since $t_1 \in (0, \infty)$, $t_1 + t_2$ can be unbounded. Therefore, the function t^2 is not uniformly continuous over $(0, \infty)$.