Indian Institute of Science
E9−252: Mathematical Methods and Techniques in Signal Processing
Instructor: Shayan G. Srinivasa
Homework #6 Solutions, Fall 2017
Solutions prepared by Priya J Nadkarni
Late submission policy: Points scored = Correct points scored × e^{-d}, d = # days late
Assigned date: Oct. 9th 2017 **Due date**: Oct. 23rd 2017 by end of the day

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PROBLEM 1:
Problem 3.19 from P.P Vaidyanathan's Book (9 points)

Solution: From the figure in the question, we obtain

$$G_m(z) = \frac{k_m^* + z^{-1}G_{m-1}(z)}{1 + k_m z^{-1}G_{m-1}(z)}$$

The poles of $G_m(z)$ are the zeros of $1 + k_m z^{-1}G_{m-1}(z)$. Thus,

$$\begin{aligned}
1 + k_m z^{-1}G_{m-1}(z) &= 0 \implies z^{-1} = -\frac{1}{k_m G_{m-1}(z)} \\
z &= -k_m G_{m-1}(z) \implies |z| = |k_m||G_{m-1}(z)| = |k_m| < 1
\end{aligned}$$

Note that z is a variable in the above case. Thus, the poles of $G_m(z)$ are inside unit circle.
As $|G_{m-1}(z)| = 1 \implies G_{m-1}(z) = \mathrm{e}^{jf(z)}$.

$$\begin{aligned}
|G_m(z)| &= \left|\frac{k_m^* + z^{-1}G_{m-1}(z)}{1 + k_m z^{-1}G_{m-1}(z)}\right| = \frac{|k_m^* + z^{-1}G_{m-1}(z)|}{|1 + k_m z^{-1}G_{m-1}(z)|} \\
\text{If } k_m = a + jb, & \\
|k_m^* + z^{-1}G_{m-1}(z)| &= |a - jb + \cos(f(z) - 1) + j\sin(f(z) - 1)| \\
&= (a^2 + \cos^2(f(z) - 1) + 2a\cos(f(z) - 1) + b^2 + \sin^2(f(z) - 1) - 2b\sin(f(z) - 1))^{1/2} \\
|1 + k_m z^{-1}G_{m-1}(z)| &= |1 + (a + jb)(\cos(f(z) - 1) + j\sin(f(z) - 1))| \\
&= (1 + a^2\cos^2(f(z) - 1) + b^2\sin^2(f(z) - 1) - 2b\sin(f(z) - 1) + 2a\cos(f(z) - 1) \\
&\quad -2ab\cos(f(z) - 1)\sin(f(z) - 1) + b^2\cos^2(f(z) - 1) + a^2\sin^2(f(z) - 1) \\
&\quad +2ab\cos(f(z) - 1)\sin(f(z) - 1))^{1/2} \\
&= (1 + a^2 + b^2 - 2b\sin(f(z) - 1) + 2a\cos(f(z) - 1))^{1/2} \\
&= |k_m^* + z^{-1}G_{m-1}(z)| \implies |G_m(z)| = 1
\end{aligned}$$

Part b) follows from induction using part a).

PROBLEM 2:
Problem 3.20 from P.P Vaidyanathan's Book (10 points)

Solution: As $P_0(z)$ is hermitian, $\tilde{P}_0(z) = z^N P_0(z)$. As $P_1(z)$ is generalized hermitian, $\tilde{P}_1(z) = cz^N P_1(z)$, where $|c| = 1$. We prove the result by expressing $A_0(z)$, $A_1(z)$ and d in terms of c, $P_0(z)$, $P_1(z)$ and $D(z)$.

$$\begin{aligned}
H_0(z) &= \frac{\beta A_0(z) + \beta^* A_1(z)}{2} \implies P_0(z) = \frac{\beta A_0(z) + \beta^* A_1(z)}{2}D(z) \\
H_1(z) &= d\frac{\beta A_0(z) - \beta^* A_1(z)}{2} \implies P_1(z) = d\frac{\beta A_0(z) - \beta^* A_1(z)}{2}D(z) \\
\implies A_0(z) &= \left(\frac{P_0(z)}{D(z)} + \frac{P_1(z)}{dD(z)}\right)\frac{1}{\beta} \text{ and } A_1(z) = \left(\frac{P_0(z)}{D(z)} - \frac{P_1(z)}{dD(z)}\right)\frac{1}{\beta^*}
\end{aligned}$$

$$|H_0(z)|^2+|H_1(z)|^2=1 \implies |P_0(z)|^2+|P_1(z)|^2=|D(z)|^2 \implies z^N P_0^2(z)+cz^N P_1^2(z)=|D(z)|^2.$$

$$\begin{aligned}
|A_0^2(z)| &= \frac{1}{|D(z)|^2}\frac{z^N}{\beta\beta^*}\left(P_0(z)+\frac{P_1(z)}{d}\right)\left(P_0(z)+c\frac{P_1(z)}{d^*}\right)\\
&= \frac{z^N}{|D(z)|^2}\left(P_0^2(z)+\frac{cP_0(z)P_1(z)}{d^*}+cP_1^2(z)+\frac{P_0(z)P_1(z)}{d}\right)\\
&= \frac{1}{|D(z)|^2}\left(|D(z)|^2+\left(\frac{c}{d^*}+\frac{1}{d}\right)P_0(z)P_1(z)\right)\\
&= 1+\left(\frac{c}{d^*}+\frac{1}{d}\right)\frac{P_0(z)P_1(z)}{|D(z)|^2}.\\
|A_1^2(z)| &= \frac{1}{|D(z)|^2}\frac{z^N}{\beta\beta^*}\left(P_0(z)-\frac{P_1(z)}{d}\right)\left(P_0(z)-c\frac{P_1(z)}{d^*}\right)\\
&= \frac{z^N}{|D(z)|^2}\left(P_0^2(z)-\frac{cP_0(z)P_1(z)}{d^*}+cP_1^2(z)-\frac{P_0(z)P_1(z)}{d}\right)\\
&= 1-\left(\frac{c}{d^*}+\frac{1}{d}\right)\frac{P_0(z)P_1(z)}{|D(z)|^2}.
\end{aligned}$$

Let us choose $d : \left(\frac{c}{d^*}+\frac{1}{d}\right)=0$, then, $|A_0^2(z)|=|A_1^2(z)|=1$. We can choose such d as $\frac{c}{d^*}=\frac{1}{d} \implies |c|=1$. Thus, we can obtain $A_0(z)$ and $A_1(z)$ as unit magnitude all pass filters with $|\beta|=|d|=1$.

PROBLEM 3:
Problem 3.21 from P.P Vaidyanathan's Book (6 points)

Solution: If $A_1(z)=cA_0(z)$, then $G(z)=(1+c)A_0(z) \implies |G(z)|=|1+c||A_0(z)|=|1+c|b$ where $|A_0(z)|=b$. Thus, $G(z)$ is all pass filter. If $G(z)=A_0(z)+A_1(z)$ is all pass filter, then, as $A_0(z)$ and $A_1(z)$ are all pass filters they can be represented by,

$$A_0(z)=be^{jf(z)} \text{ and } A_1(z)=ce^{jh(z)}$$

$G(z)$ is obtained as:

$$\begin{aligned}
G(z) &= A_0(z)+A_1(z)=be^{jf(z)}+ce^{jh(z)}\\
&= b\cos f(z)+jb\sin f(z)+c\cos h(z)+jc\sin h(z)\\
|G(z)|^2 &= b^2\cos^2 f(z)+c^2\cos^2 h(z)+2bc\cos f(z)\cos h(z)+b^2\sin^2 f(z)+c^2\sin^2 h(z)+2bc\sin f(z)\sin h(z)\\
&= b^2+c^2+2bc\cos(f(z)-h(z))
\end{aligned}$$

As b, c and $|G(z)|^2$ are constant, thus,

$$\begin{aligned}
\cos(f(z)-h(z)) &= d \text{ (const)} \implies f(z)-h(z)=\cos^{-1}d=d' \implies h(z)=f(z)-d'\\
\implies A_1(z) &= ce^{jh(z)}=ce^{jf(z)-d'}=mA_0(z).
\end{aligned}$$