Indian Institute of Science

E9−252: Mathematical Methods and Techniques in Signal Processing

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Homework #2 Solutions, Fall 2017

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Late submission policy: Points scored = Correct points scored × e^{-d}, d = # days late

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PROBLEM 1:

If $x(t) = \sum_{k=1}^{M} A_k e^{j2\pi f_k t}$, $E[A_k] = 0$ and A_k's are uncorrelated, examine if $x(t)$ is WSS.

Solution: For a process to be WSS, we need to check two conditions:

1) $E[x(t)]$ should be a constant with respect to time t. Let us check it for our signal.

$$E[x(t)] = \sum_{k=1}^{M} \mathrm{E}[A_k e^{j2\pi f_k t}] = \sum_{k=1}^{M} \mathrm{E}[A_k]\mathrm{E}[e^{j2\pi f_k t}] = 0$$

2) $R_{xx}(t_1, t_2)$ depends on only the time difference $t_1 - t_2$.

$$\begin{aligned} R_{xx}(t_1,t_2) &= \mathrm{E}[x(t_1)x^*(t_2)] = \mathrm{E}\left[\sum_{k=1}^{M} A_k e^{j2\pi f_k t_1} \sum_{l=1}^{M} A_l^* e^{-j2\pi f_l t_2}\right] \\ &= \sum_{k=1}^{M}\sum_{l=1}^{M} \mathrm{E}[A_k A_l^*]\mathrm{E}[e^{j2\pi(f_k t_1 - f_l t_2)}] \end{aligned}$$

As A_k's are uncorrelated, if $k \neq l$, $\mathrm{E}[A_k A_l^*] = \mathrm{E}[A_k]\mathrm{E}[A_l^*] = \mathrm{E}[A_k]\mathrm{E}[A_l]^* = 0$. Thus,

$$R_{xx}(t_1,t_2) = \sum_{k=1}^{M} \mathrm{E}[|A_k|^2]\mathrm{E}[e^{j2\pi(f_k t_1 - f_k t_2)}] = \sum_{k=1}^{M} \mathrm{E}[|A_k|^2]\mathrm{E}[e^{j2\pi f_k (t_1 - t_2)}]$$

Thus, this process is WSS.

PROBLEM 2:

Prove the following:

a) $|R_{XX}(\tau)| \leq R_{XX}(0)$

b) $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$

c) $R_{XX}(\tau) = R_{XX}^*(-\tau)$

d) $\sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l^* R_{XX}(t_k - t_l) \geq 0 \quad \forall N > 0,\ \forall t_1 < t_2 < \cdots < t_N$ and complex a_i's

Solution: Let us solve part b) first.

b)

$$
\begin{aligned}
\mathrm{E}[|x(t) - \alpha y(t-\tau)|^2] &\geq 0 \\
\mathrm{E}[|x(t)|^2] + |\alpha|^2 |y(t-\tau)|^2 - \alpha^* x(t) y^*(t-\tau) - \alpha x^*(t) y(t-\tau)] &\geq 0 \\
R_{xx}(0) + |\alpha|^2 R_{yy}(0) - \alpha^* R_{xy}(\tau) - \alpha R^*_{xy}(\tau) &\geq 0 \\
\text{differentiating w.r.t } \alpha^*, \quad \alpha R_{yy}(0) - R_{xy}(\tau) = 0 &\Rightarrow \alpha = \frac{R_{xy}(\tau)}{R_{yy}(0)} \\
\text{Thus, } R_{xx}(0) + |\frac{R_{xy}(\tau)}{R_{yy}(0)}|^2 R_{yy}(0) - \frac{R_{xy}(\tau)}{R_{yy}(0)} R^*_{xy}(\tau) - \frac{R^*_{xy}(\tau)}{R^*_{yy}(0)} R_{xy}(\tau) &\geq 0 \\
R_{xx}(0) R_{yy}(0) &\geq |R_{xy}(\tau)|^2 \\
|R_{xy}(\tau)| &\leq \sqrt{R_{xx}(0) R_{yy}(0)}
\end{aligned}
$$

a) This result is obtained by substituting $y = x$ in part b)

c)

$$
\begin{aligned}
R^*_{xx}(-\tau) &= \mathrm{E}[x(t) x^*(t-\tau)]^* = \mathrm{E}[x^*(t) x(t-\tau)] \\
&= \mathrm{E}[x(t) x^*(t-\tau)] = R_{xx}(\tau)
\end{aligned}
$$

d) Let $x(t)$ denote a WSS process. Consider $y(t) = \sum_{k=1}^{N} a_k x(t_k - t)$.

$$
\begin{aligned}
\mathrm{E}[|y(t)|^2] \geq 0 \Rightarrow \mathrm{E}[\sum_{k=1}^{N} a_k x(t_k - t) \sum_{l=1}^{N} a_l^* x^*(t_l - t)] &\geq 0 \\
\sum_{k=1}^{N} \sum_{l=1}^{N} a_k a_l^* \mathrm{E}[x(t_k - t) x^*(t_l - t)] &\geq 0 \\
\sum_{k=1}^{N} \sum_{l=1}^{N} a_k a_l^* R_{xx}(t_k - t_l) &\geq 0
\end{aligned}
$$

PROBLEM 3:

a) Only one of the switches S_1, S_2 and S_3 is active at a time. S_1 closes twice as fast as S_2. S_2 closes twice as fast as S_3.The signals are distributed normally as follows:

$$A \sim \mathcal{N}(-1, 4), B \sim \mathcal{N}(0, 1) \text{ and } C \sim \mathcal{N}(1, 4)$$

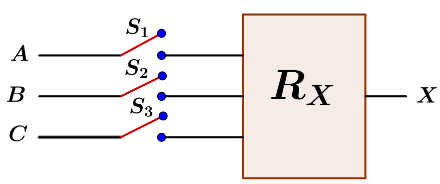

Figure 1: Switch

i) What is $P(X \leq 1)$?

ii) Given $X > -1$, which signal is most likely transmitted?

b) There are two roads from A to B and two roads from B to C. Each of the four roads have probability p of being blocked by snow independently of all the others. What is the probability of an open road from A to C?

Solution:

a) $P(X \leq 1) = \sum_{i=1}^{3} P(X \leq 1|S_i \text{ is active})P(S_i \text{ is active})$.

$$P(X \leq 1|S_i \text{ is active}) = \begin{cases} P(A \leq 1) & i = 1 \\ P(B \leq 1) & i = 2 \\ P(C \leq 1) & i = 3 \end{cases} \tag{1}$$

Computing the CDF for a variable M having normal distribution $\mathcal{N}(\mu, \sigma^2)$:

$$P(M \leq b) = \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(m-\mu)^2}{2\sigma^2}} dm$$

Considering $y = \frac{m-\mu}{\sigma}$, we obtain $dy = \frac{dm}{\sigma}$ and limits change to $-\infty$ and $b' = \frac{b-\mu}{\sigma}$

$$P(M \leq b) = \int_{-\infty}^{b'} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = P(Y \leq b' = \frac{b-\mu}{\sigma}) \text{ where } Y = \frac{M-\mu}{\sigma}$$

Thus, let Y be a random variable having standard normal distribution $\mathcal{N}(\mu, \sigma^2)$.

$$\begin{aligned} P(A \leq 1) &= P(Y \leq (1-(-1))/2) = P(X \leq 1) = 0.8413 \\ P(B \leq 1) &= P(Y \leq (1-0)/1) = P(X \leq 1) = 0.8413 \\ P(C \leq 1) &= P(Y \leq (1-(1))/2) = P(X \leq 0) = 0.5 \end{aligned}$$

Similarly, for (ii),

$$\begin{aligned} P(A \leq -1) &= P(Y \leq (-1-(-1))/2) = P(X \leq 0) = 0.5 \Rightarrow P(A > -1) = 0.5 \\ P(B \leq -1) &= P(Y \leq (-1-0)/1) = P(X \leq -1) = 0.1587 \Rightarrow P(B > -1) = 0.8413 \\ P(C \leq -1) &= P(Y \leq (-1-(1))/2) = P(X \leq -1) = 0.1587 \Rightarrow P(C > -1) = 0.8413 \end{aligned}$$

The CDF for standard normal distribution is obtained from the table.
Now, $P(S_1 \text{ is active}) : P(S_2 \text{ is active}) : P(S_3 \text{ is active}) = 4 : 2 : 1$
$\Rightarrow P(S_1 \text{ is active}) = 4/7, P(S_2 \text{ is active}) = 2/7$ and $P(S_3 \text{ is active}) = 1/7$.

$$\begin{aligned} P(X \leq 1) &= (4/7) \times 0.8413 + (2/7) \times 0.8413 + (1/7) \times 0.5 = 0.7925 \\ P(X \leq -1) &= (4/7) \times 0.5 + (2/7) \times 0.1587 + (1/7) \times 0.1587 = 0.3537 \Rightarrow P(X > -1) = 0.6463 \end{aligned}$$

When $X > -1$, the signal which was most likely to be transmitted is computed based on aposteriori probability,

$$\begin{aligned} P(X = A|X > -1) &= \frac{P(X > -1|X = A)P(X = A)}{P(X > -1)} = \frac{0.5 \times \frac{4}{7}}{0.6463} = 0.4421 \\ P(X = B|X > -1) &= \frac{P(X > -1|X = B)P(X = B)}{P(X > -1)} = \frac{0.8413 \times \frac{2}{7}}{0.6463} = 0.3719 \\ P(X = C|X > -1) &= \frac{P(X > -1|X = C)P(X = C)}{P(X > -1)} = \frac{0.8413 \times \frac{1}{7}}{0.6463} = 0.1859 \end{aligned}$$

Thus, the most likely transmitted signal is A.

b) Let r_1 and r_2 be roads from A to B and r_3 and r_4 be roads from B to C. The probability that a road is not blocked is $1-p$. Thus,

$$\begin{aligned}
\text{P(Road open from A to C)} &= \text{P(Road open from A to B)P(Road open from B to C)} \\
\text{P(Road open from A to B)} &= \text{P(Road open from A to B)} \qquad \text{due to symmetry} \\
\text{P(Road open from A to B)} &= \text{P}(r_1 \text{ or } r_1 \text{ is open}) \\
&= p(1-p) + (1-p)p + (1-p)^2 = 1-p^2 \\
\text{P(Road open from A to C)} &= (1-p^2)^2
\end{aligned}$$

PROBLEM 4: Prove the Cauchy Schwarz inequality for random variables: For two random variables X and Y,

$$|\text{Cov}(X,Y)| \leq \sqrt{\text{Var}(X)\text{Var}(Y)}.$$

Solution: Let X and Y be two random variables. Let us convert them to random variables A and B which have zero mean and variance equal to 1.

$$A = \frac{X - \text{E}[X]}{\sigma_X}, \quad B = \frac{Y - \text{E}[Y]}{\sigma_Y}$$

Now, as $\text{E}[(A+B)^2] \geq 0$ and $\text{E}[(A-B)^2] \geq 0$, we have,

$$\begin{aligned}
\text{E}[A^2 + B^2 + 2AB] &\geq 0 \Rightarrow \text{E}[A^2] + \text{E}[B^2] + 2\text{E}[AB] \geq 0 \\
\text{E}[AB] &\geq (-\sigma_A - \sigma_B)/2 = -1 \\
\text{E}[A^2 + B^2 - 2AB] &\geq 0 \Rightarrow \text{E}[A^2] + \text{E}[B^2] - 2\text{E}[AB] \geq 0 \\
\text{E}[AB] &\leq (\sigma_A + \sigma_B)/2 = 1 \\
\Rightarrow |\text{E}[AB]| \leq 1
\end{aligned}$$

Equality occurs when $\text{E}[(A+B)^2] = 0$ or $\text{E}[(A-B)^2] = 0$, i.e. when $A = -B$ or $A = B$. Considering X and Y,

$$\begin{aligned}
|\text{Cov}(X,Y)| &= |E[(X - E[X])(Y - E[Y])]| = |E[\sigma_X A \sigma_Y B]| \\
&= |\sigma_X \sigma_Y E[AB]| = |\sigma_X \sigma_Y||E[AB]| \\
&\leq \sqrt{\sigma_X^2 \sigma_Y^2} \\
&= \sqrt{\text{Var}(X)\text{Var}(Y)}
\end{aligned}$$