Indian Institute of Science

E9−252: Mathematical Methods and Techniques in Signal Processing
Instructor: Shayan G. Srinivasa
Homework #0 Solutions, Fall 2017
Late submission policy: Points scored = Correct points scored × e^{-d}, d = # days late
Assigned date: Aug. 21st 2017 **Due date**: Aug. 28th 2017 by end of the day

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PROBLEM 1: (Linearity)

a) Check if the $f(x) = \log_2(\cosh x + \sinh x)^3$ is a linear function. (2 Points)

b) Examine if the composition of two linear maps is linear. (3 Points)

Solution:

A function $f : \mathrm{X} \to \mathrm{Y}$ is said to be linear if for every $x_1, x_2 \in \mathrm{X}$ and constants a and b, the function satisfies $f(ax_1 + bx_2) = af(x_1) + bf(x_2)$.

a) $f(x)$ can be simplified as follows:

$$\begin{aligned} f(x) &= \log_2(\cosh x + \sinh x)^3 \\ &= \log_2\left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right)^3 \\ &= \log_2(\mathrm{e}^x)^3 \\ &= 3x\log_2\mathrm{e} \end{aligned} \tag{1}$$

Test for linearity:

$$\begin{aligned} f(ax_1 + bx_2) &= 3(ax_1 + bx_2)\log_2\mathrm{e} \\ &= a(3x_1\log_2\mathrm{e}) + b(3x_2\log_2\mathrm{e}) \\ &= af(x_1) + bf(x_2) \quad \text{(from eq. 1)} \end{aligned}$$

Thus, the function $f(x)$ is linear.

b) The composition of two linear maps $f : \mathrm{X} \to \mathrm{Y}$ and $g : \mathrm{Y} \to \mathrm{Z}$ is given by:

$$(g \circ f)(\cdot) = g(f(\cdot))$$

Test for linearity:

$$\begin{aligned} (g \circ f)(ax_1 + bx_2) &= g(f(ax_1 + bx_2)) \\ &= g(af(x_1) + bf(x_2)) \quad \text{(as } f(\cdot) \text{ is linear)} \\ &= ag(f(x_1)) + bg(f(x_2)) \quad \text{(as } g(\cdot) \text{ is linear)} \\ &= a(g \circ f)(x_1) + b(g \circ f)(x_2) \end{aligned}$$

Thus, composition of two linear maps is linear.

PROBLEM 2:
Solve problem 1.4.16 and 1.4.18(c) from Moon and Stirling Book. (5 Points)
Note: Problem 1.4.16 will not be graded.

Solution 1.4.16: The transfer function of the system with $\overline{A} = T^{-1}AT$, $\overline{b} = T^{-1}b$, $\overline{c} = T^{\mathrm{T}}c$, $\overline{d} = d$ is,

$$
\begin{aligned}
\overline{H}(z) &= \overline{c}^{\mathrm{T}}(z\mathrm{I} - \overline{A})^{-1}\overline{b} + \overline{d} \\
&= (T^{\mathrm{T}}c)^{\mathrm{T}}(z\mathrm{I} - T^{-1}AT)^{-1}T^{-1}b + d \\
&= c^{\mathrm{T}}T(zT^{-1}T - T^{-1}AT)^{-1}T^{-1}b + d \\
&= c^{\mathrm{T}}T(T^{-1}(z\mathrm{I} - A)T)^{-1}T^{-1}b + d \\
&= c^{\mathrm{T}}TT^{-1}(z\mathrm{I} - A)^{-1}TT^{-1}b + d \quad (as (LK)^{-1} = K^{-1}L^{-1}) \\
&= c^{\mathrm{T}}(z\mathrm{I} - A)^{-1}b + d \\
&= H(z)
\end{aligned}
$$

Solution 1.4.18(c):

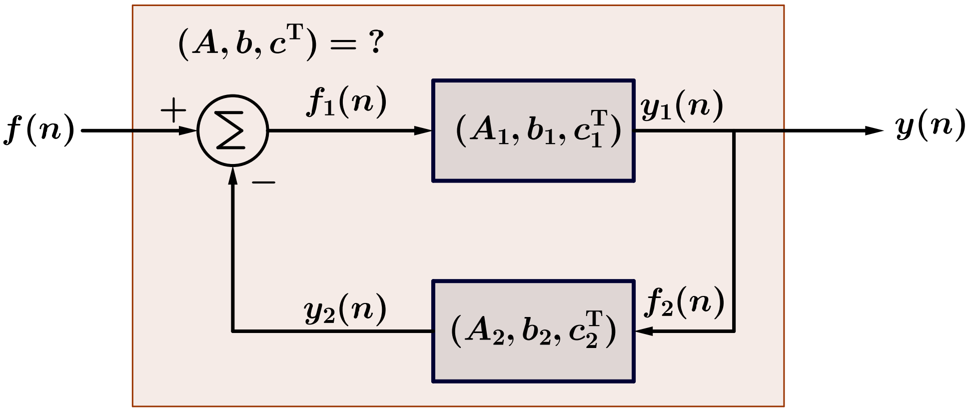

Figure 1: Feedback System

For the forward path,

$$
\begin{aligned}
x_1(n+1) &= A_1x_1(n) + b_1f_1(n) \\
y_1(n) &= c_1^{\mathrm{T}}x_1(n)
\end{aligned}
\tag{2}
$$

For the backward path,

$$
\begin{aligned}
x_2(n+1) &= A_2x_2(n) + b_2f_2(n) \\
y_2(n) &= c_2^{\mathrm{T}}x_2(n)
\end{aligned}
$$

The output of the system is,

$$
y(n) = y_1(n) = c_1^{\mathrm{T}}x_1(n) \tag{3}
$$

The input of the system is,

$$
f_1(n) = f(n) - c_2^{\mathrm{T}}x_2(n) \tag{4}
$$

Substituting eq. 4 in eq. 2,

$$
\begin{aligned}
x_1(n+1) &= A_1x_1(n) + b_1(f(n) - c_2^{\mathrm{T}}x_2(n)) \\
&= A_1x_1(n) + b_1f(n) - b_1c_2^{\mathrm{T}}x_2(n)
\end{aligned}
\tag{5}
$$

As $f_2(n) = y(n)$ and from eq. 3,

$$x_2(n+1) = A_2 x_2(n) + b_2 c_1^{\mathrm{T}} x_1(n) \tag{6}$$

Considering the state of the system as $\begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$, we obtain,

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} A_1 & -b_1 c_2^{\mathrm{T}} \\ b_2 c_1^{\mathrm{T}} & A_2 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} f(n) \tag{7}$$

and the output of the system is,

$$y = \begin{bmatrix} c_1^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} \tag{8}$$

From eq. 7 and 8,

$$A = \begin{bmatrix} A & -b_1 c_2^{\mathrm{T}} \\ b_2 c_1^{\mathrm{T}} & A_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \text{ and } c^{\mathrm{T}} = \begin{bmatrix} c_1^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}}$$

PROBLEM 3:
Obtain the steady state output and the state space representation for system with input $x(n) = (\frac{1}{2})^{\mathrm{n}} \mathrm{u}(n)$ and transfer function (10 = 5+2+3 Points)

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}.$$

Solution: From $H(z)$, $a_0 = b_0 = 1, a_1 = -0.75, a_2 = 0.125, b_1 = 2$ and $b_2 = 1$. Thus, the state space representation is,

$$A = \begin{bmatrix} 0 & 1 \\ -0.125 & 0.75 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0.875 \\ 2.75 \end{bmatrix} \quad \text{and } d = b_0 = 1$$

To compute the steady state output,

$$\begin{aligned} Y(z) &= H(z)X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} \mathcal{Z}\left\{ \left(\frac{1}{2}\right)^n \mathrm{u}(n) \right\} \\ &= \frac{z^2 + 2z + 1}{z^2 - 0.75z + 0.125} \frac{1}{1 - \frac{1}{2}z^{-1}} \\ \frac{Y(z)}{z} &= \frac{z^2 + 2z + 1}{(z - \frac{1}{2})^2 (z - \frac{1}{4})} = \frac{-24}{z - \frac{1}{2}} + \frac{9}{(z - \frac{1}{2})^2} + \frac{25}{z - \frac{1}{4}} \\ Y(z) &= \frac{-24z}{z - \frac{1}{2}} + \frac{9z}{(z - \frac{1}{2})^2} + \frac{25z}{z - \frac{1}{4}} \end{aligned}$$

Using inverse Z-transform,

$$\begin{aligned} y(n) &= -24 \left(\frac{1}{2}\right)^n \mathrm{u}(n) + 18n \left(\frac{1}{2}\right)^n \mathrm{u}(n) + 25 \left(\frac{1}{4}\right)^n \mathrm{u}(n) \\ &= \left(6 \left(\frac{1}{2}\right)^n (3n - 4) + 25 \left(\frac{1}{4}\right)^n \right) \mathrm{u}(n) \end{aligned} \tag{9}$$

The steady state output is,

$$\lim_{n\to\infty} y(n) \;=\; \lim_{n\to\infty}\left(6\left(\frac{1}{2}\right)^n(3n-4)+25\left(\frac{1}{4}\right)^n\right)\mathrm{u}(n)=0.$$

Code snippet to simulate the system using Matlab:

```
clc;
A = [0 1; −0.125 0.75]; b = [0 1]'; c = [0.875 2.75]; d = 1;
n = 100; x1 = 0; x2 = 0;
y1 = zeros(1, n); y2 = zeros(1, n);
for i = 0 : n
  f = (0.5)^i;
  y1(i + 1) = c ∗ [x1 x2]' + d ∗ f;
  t = A ∗ [x1 x2]' + b ∗ f;
  x1 = t(1); x2 = t(2);
  y2(i + 1) = 6 ∗ (0.5)^i ∗ (3 ∗ i − 4) + 25 ∗ (0.25)^i;
end
figure(1);
subplot(2,6,1:3); P11 = plot(y1);set(P11,'Color','blue');
xlabel('n'); ylabel('y1(n)');
subplot(2,6,4:6); P11 = plot(y2);set(P11,'Color','red');
xlabel('n'); ylabel('y2(n)');
subplot(2,6,8:11); P1 = plot(y1); set(P1,'Color','blue');
hold on; P2 = plot(y2); set(P2,'Color','red');
xlabel('n'); ylabel('y1(n) (Blue) and y2(n) (Red)');
```

Simulation Results:

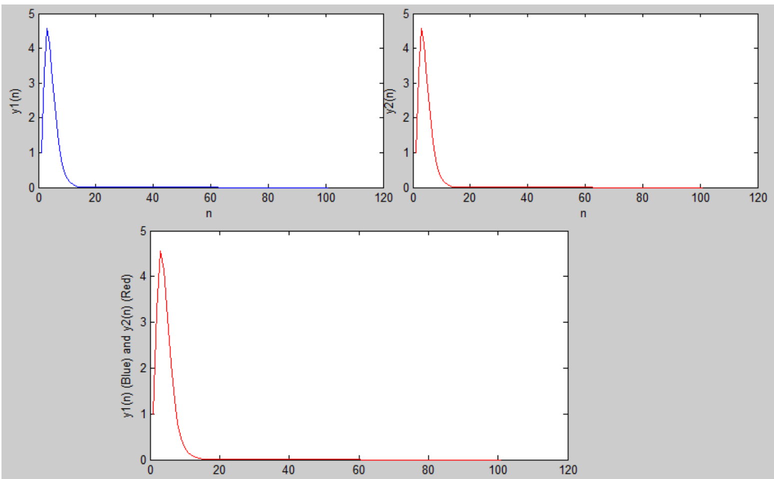

Figure 2: Comparison of results

$y1$ denotes the output response obtained by using the state space representation. $y2$ denotes the output response obtained using the time domain expression in eq. 9. $y1$ and $y2$ are plotted in 1st and 2nd subplot respectively. In the last subplot, both are plotted together. We observe that they both overlap and hence are the same.

PROBLEM 4: (System Modes)
Calculate the number of system modes with impulse response of the system $y(n) = \left\{1, \frac{3}{4}, \frac{1}{2}, \frac{5}{16}, \ldots\right\}$. (5 Points)
Solution: The impulse response $y(n) = \left\{1, \frac{3}{4}, \frac{1}{2}, \frac{5}{16}, \ldots\right\}$, can be written as:

$$y(n) = \frac{n+2}{2^{n+1}}\mathrm{u}(n)$$

The $Z-$transform of $y(n)$ is given by,

$$\begin{aligned}
Y(z) &= \mathcal{Z}\left\{\frac{n+2}{2^{n+1}}\right\} \\
&= \mathcal{Z}\left\{\frac{n}{2^{n+1}}\right\} + \mathcal{Z}\left\{\frac{2}{2^{n+1}}\right\} \\
&= -z\frac{\mathrm{d}}{\mathrm{dz}}\left(\frac{1}{2(1-\frac{z^{-1}}{2})}\right) + \frac{1}{(1-\frac{z^{-1}}{2})} \\
&= -z\frac{\mathrm{d}}{\mathrm{dz}}\left(\frac{z}{2(z-\frac{1}{2})}\right) + \frac{z}{z-\frac{1}{2}} \\
&= \frac{z}{4(z-\frac{1}{2})^2} + \frac{z}{z-\frac{1}{2}} \\
&= \frac{z(z-\frac{1}{4})}{(z-\frac{1}{2})^2}
\end{aligned}$$

The modes of the system are obtained from the poles. Thus, we have two modes for the system namely $\frac{1}{2}$ and $\frac{1}{2}$.