INDIAN INSTITUTE OF SCIENCE
E9-252: MATHEMATICAL METHODS AND TECHNIQUES IN SIGNAL PROCESSING
HOME WORK #2 - SOLUTIONS, FALL 2015

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Problem 1. P. P. Vaidyanathan, Problem #4.1. (6 pts)

Solution. The output signals $Y_0\left(e^{j\omega}\right)$ and $Y_1\left(e^{j\omega}\right)$ are as shown in Figure 1.

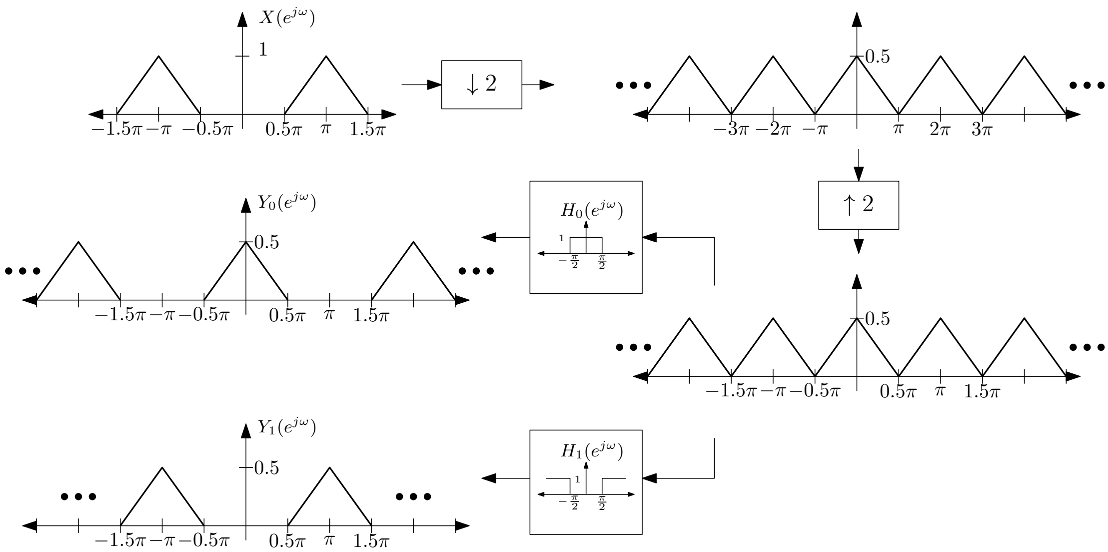

FIGURE 1. Frequency domain representation of signal at various stages in the given transformation.

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Problem 2. P. P. Vaidyanathan, Problem #4.9. (7 pts)

Solution. At time t, the painter paints the $tM \mod N$ object, $t = 0, 1, 2, \cdots$.

For $t \geq N$, $tM \mod N = (tM - MN) \mod N = ((t-N)M) \mod N$.

Therefore, it is enough to check if the painter paints all objects in the first N instants i.e., for $t = 0, 1, \cdots, N-1$.

Essentially, we need a condition that

$$\{tM \mod N \mid t = 0, 1, \cdots, N-1\} = \{0, 1, \cdots, N-1\}$$

Without loss of generality, we can consider $1 \leq M \leq N$. If $M > N$, we can consider $1 \leq M' = M - kN \leq N$ and hence $kM \mod N = kM' \mod N \, \forall k$.

Necessary condition:

The case under which the painter does not paint all objects is when $t_1 M \mod N = t_2 M \mod N$ for some $t_1 \neq t_2$ and $0 \leq t_1 < t_2 \leq N-1$.

$$\begin{aligned} t_1 M \mod N &= t_2 M \mod N \\ \implies (t_2 - t_1) M \mod N &= 0 \end{aligned}$$

Since $t_2 - t_1 \neq 0$ and $t_1 - t_2 \leq N - 1$, N does not divide $t_2 - t_1$. Therefore, $(t_1 - t_2) M \mod N = 0$ holds true only if some factor($\neq 1$) of N divides M i.e., $gcd(M, N) \neq 1$.

The necessary condition for the painter to paint all objects is that M and N are co-prime.

Sufficient condition:

Let $gcd(M, N) = 1$. Since $0 < t_2 - t_1 \leq N - 1$ for all $t_1 \neq t_2$, $0 \leq t_1 < t_2 \leq N-1$, N does not divide $t_2 - t_1$. Therefore, N does not divide $(t_2 - t_1) M$ as $gcd(M, N) = 1$. Therefore,

$$\begin{aligned} (t_2 - t_1) M \mod N &\neq 0 \\ t_2 M \mod N &\neq t_1 M \mod N, \quad 0 \leq t_1 < t_2 \leq N-1. \end{aligned}$$

Hence, the elements $tM \mod N$, $0 \leq t \leq N-1$ are all unique. Therefore,

$$\{tM \mod N \mid t = 0, 1, \cdots, N-1\} = \{0, 1, \cdots, N-1\}.$$

Therefore, the painter paints all objects.

Hence, the necessary and sufficient condition for the painter to paint all objects is $gcd(M, N) = 1$. ■

Problem 3. P. P. Vaidyanathan, Problem #4.13. (5 pts)

Solution. We have

$$\begin{aligned}
H(z) &= \frac{a+z^{-1}}{1+az^{-1}} \\
&= \frac{\left(a+z^{-1}\right)\left(1-az^{-1}\right)}{1-a^2z^{-2}} \\
&= \frac{a\left(1-z^{-2}\right)+\left(1-a^2\right)z^{-2}}{1-a^2z^{-2}} \\
&= \underbrace{\frac{a\left(1-z^{-2}\right)}{1-a^2z^{-2}}}_{E_0(z^2)} + \underbrace{\frac{1-a^2}{1-a^2z^{-2}}}_{E_1(z^2)} z^{-1} \\
H(z) &= \sum_{i=0}^{1} z^{-i} E_i\left(z^2\right) \\
\text{where, } E_0(z) &= \frac{a\left(1-z^{-1}\right)}{1-a^2z^{-1}}, \\
E_1(z) &= \frac{1-a^2}{1-a^2z^{-1}}.
\end{aligned}$$

This gives the type-I decomposition of $H(z)$.

$\left|E_0\left(e^{j\times 0}\right)\right| = \begin{cases} 1, & a=1 \\ 0, & a\neq 1 \end{cases}$, $\left|E_0\left(e^{j\pi}\right)\right| = \left|\frac{2a}{1+a^2}\right|$. Therefore, $E_0(z)$ is all-pass filter for $a=1$. For $a\neq 1$, $E_0(z)$ is not an all-pass filter.

$\left|E_1\left(e^{j\times 0}\right)\right| = 1$, $\left|E_1\left(e^{j\pi}\right)\right| = \frac{1-a^2}{1+a^2}$. Therefore, $E_0(z)$ is not an all-pass filter. ■

Problem 4. P. P. Vaidyanathan, Problem #4.18. (7 pts)

Solution. For $M = 4$, the DFT filter bank is given in Figure 2.

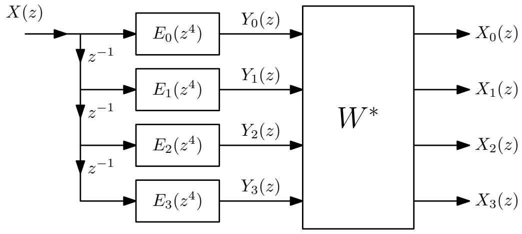

FIGURE 2. DFT filter bank.

The inputs to the DFT block are

$$Y_k(z) = z^{-k} E_k\left(z^M\right) X(z), \quad k = 0, 1, \cdots, M-1.$$

The outputs of the DFT filter bank are

$$\begin{aligned} X_k(z) &= \sum_{i=0}^{M-1} \omega^{-ik} Y_i(z), \quad , k = 0, 1, \cdots, M-1 \\ &= \sum_{i=0}^{M-1} \omega^{-ik} z^{-i} E_i\left(z^M\right) X(z), \\ \text{where } \omega &= e^{-j\frac{\pi}{2}} = -j. \end{aligned}$$

Therefore,

$$H_k(z) = \frac{X_k(z)}{X(z)} = \sum_{i=0}^{M-1} \omega^{-ik} z^{-i} E_i\left(z^M\right)$$

$$\begin{aligned} H_0(z) &= \sum_{i=0}^{4} z^{-i} E_i\left(z^4\right) \\ &= \left(1 + z^{-4}\right) + z^{-1}\left(1 + 2z^{-4}\right) + z^{-2}\left(2 + z^{-8}\right) + z^{-3}\left(0.5 + z^{-4}\right) \\ H_0(z) &= 1 + z^{-1} + 2z^{-2} + 0.5z^{-3} + z^{-4} + 2z^{-5} + z^{-7} + z^{-10} \end{aligned}$$

$$\begin{aligned} H_1(z) &= \sum_{i=0}^{4} (-j)^{-i} z^{-i} E_i\left(z^4\right) \\ &= \left(1 + z^{-4}\right) + jz^{-1}\left(1 + 2z^{-4}\right) - z^{-2}\left(2 + z^{-8}\right) - jz^{-3}\left(0.5 + z^{-4}\right) \\ H_1(z) &= 1 + jz^{-1} - 2z^{-2} - 0.5jz^{-3} + z^{-4} + 2jz^{-5} - jz^{-7} - z^{-10} \end{aligned}$$

$$\begin{aligned} H_2(z) &= \sum_{i=0}^{4} (-1)^{-i} E_i\left(z^4\right) \\ &= \left(1 + z^{-4}\right) - z^{-1}\left(1 + 2z^{-4}\right) + z^{-2}\left(2 + z^{-8}\right) - z^{-3}\left(0.5 + z^{-4}\right) \\ H_2(z) &= 1 - z^{-1} + 2z^{-2} - 0.5z^{-3} + z^{-4} - 2z^{-5} - z^{-7} + z^{-10} \end{aligned}$$

$$\begin{aligned} H_3(z) &= \sum_{i=0}^{4} j^{-i} z^{-i} E_i\left(z^4\right) \\ &= \left(1 + z^{-4}\right) - jz^{-1}\left(1 + 2z^{-4}\right) - z^{-2}\left(2 + z^{-8}\right) + jz^{-3}\left(0.5 + z^{-4}\right) \\ H_3(z) &= 1 - jz^{-1} - 2z^{-2} + 0.5jz^{-3} + z^{-4} - 2jz^{-5} + jz^{-7} - z^{-10} \end{aligned}$$

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Problem 5. P. P. Vaidyanathan, Problem #4.19. (10 pts)

Solution. The DFT synthesis structure is given in Figure 3.

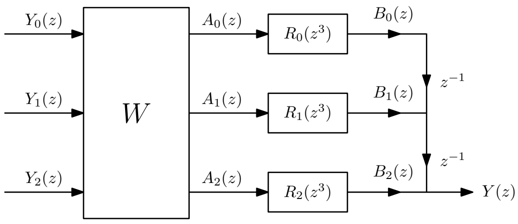

FIGURE 3. 3 channel DFT synthesis filters.

The outputs of the DFT block are

$$\begin{aligned} A_k(z) &= \sum_{i=0}^{2} \omega_3^{ik} Y_i(z), \quad k = 0,1,2 \\ \text{where } \omega_3 &= e^{-j\frac{2\pi}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} j. \end{aligned}$$

$$B_k(z) = R_k\left(z^3\right) A_k(z), \quad k = 0,1,2.$$

$$\begin{aligned} Y(z) &= \sum_{k=0}^{2} z^{k-2} B_k(z) \\ &= \sum_{k=0}^{2} z^{k-2} R_k\left(z^3\right) A_k(z) \\ Y(z) &= \sum_{k=0}^{2} \left(z^{k-2} R_k\left(z^3\right) \sum_{i=0}^{2} \omega_3^{ik} Y_i(z) \right) \end{aligned}$$

We obtain $F_l(z) = \frac{Y(z)}{Y_l(z)}$ by setting $Y_i(z) = 0 \, \forall i \neq l$. In this case,

$$\begin{aligned} Y(z) &= \sum_{k=0}^{2} \left(z^{k-2} R_k\left(z^3\right) \omega_3^{lk} Y_l(z) \right) \\ &= Y_l(z) \sum_{k=0}^{2} R_k\left(z^3\right) \\ \Longrightarrow F_l(z) &= \sum_{k=0}^{2} z^{k-2} \omega_3^{lk} R_k\left(z^3\right) \end{aligned} \tag{1}$$

$$\begin{aligned} F_0(z) &= \sum_{k=0}^{2} z^{k-2} R_k\left(z^3\right) \\ &= z^{-2}\left(1 + z^{-3}\right) + z^{-1}\left(1 - z^{-6}\right) + \left(2 + 3z^{-3}\right) \\ F_0(z) &= 2 + z^{-1} + z^{-2} + 3z^{-3} + z^{-5} - z^{-7} \end{aligned}$$

$$\begin{aligned} F_1(z) &= \sum_{k=0}^{2} z^{k-2} \omega_3^{k} R_k\left(z^3\right) \\ &= z^{-2}\left(1 + z^{-3}\right) + \omega_3 z^{-1}\left(1 - z^{-6}\right) + \omega_3^2\left(2 + 3z^{-3}\right) \\ &= 2\omega_3^2 + \omega_3 z^{-1} + z^{-2} + 3\omega_3^2 z^{-3} + z^{-5} - \omega_3 z^{-7} \\ F_1(z) &= \left(-1 + \sqrt{3}j\right) + \frac{\left(-1 - \sqrt{3}j\right)}{2} z^{-1} + z^{-2} + 3\frac{\left(-1 + \sqrt{3}j\right)}{2} z^{-3} + z^{-5} - \frac{\left(-1 - \sqrt{3}j\right)}{2} z^{-7} \end{aligned}$$

$$F_2(z) = \sum_{k=0}^{2} z^{k-2} \omega_3^{2k} R_k\left(z^3\right)$$

$$
\begin{aligned}
&= z^{-2}\left(1+z^{-3}\right)+\omega_3^2 z^{-1}\left(1-z^{-6}\right)+\omega_3\left(2+3z^{-3}\right)\\
&= 2\omega_3+\omega_3^2 z^{-1}+z^{-2}+3\omega_3 z^{-3}+z^{-5}-\omega_3^2 z^{-7}\\
F_2(z) &= \left(-1-\sqrt{3}j\right)+\frac{\left(-1+\sqrt{3}j\right)}{2}z^{-1}+z^{-2}+3\frac{\left(-1-\sqrt{3}j\right)}{2}z^{-3}+z^{-5}-\frac{\left(-1+\sqrt{3}j\right)}{2}z^{-7}.
\end{aligned}
$$

From (1), we have

$$
\begin{aligned}
F_0(z) &= \sum_{k=0}^{2} z^{k-2}R_k\left(z^3\right)\\
&= z^{-2}\sum_{k=0}^{2} z^{k}R_k\left(z^3\right)\\
\Longrightarrow \left|F_0\left(e^{j\omega}\right)\right| &= \left|e^{-2j\omega}\sum_{k=0}^{2} e^{j\omega k}R_k\left(e^{3j\omega}\right)\right|\\
\left|F_0\left(e^{j\omega}\right)\right| &= \left|\sum_{k=0}^{2} e^{j\omega k}R_k\left(e^{3j\omega}\right)\right|\\
F_l(z) &= \sum_{k=0}^{2} z^{k-2}\omega_3^{lk}R_k\left(z^3\right)\\
&= z^{-2}\sum_{k=0}^{2}\left(z\omega_3^l\right)^k R_k\left(\left(z\omega_3^l\right)^3\right)\\
\Longrightarrow \left|F_l\left(e^{j\omega}\right)\right| &= \left|e^{-2j\omega}\sum_{k=0}^{2} e^{j\left(\omega-l\frac{2\pi}{3}\right)k}R_k\left(e^{j3\left(\omega-l\frac{2\pi}{3}\right)k}\right)\right|\\
&= \left|\sum_{k=0}^{2} e^{j\left(\omega-l\frac{2\pi}{3}\right)k}R_k\left(e^{j3\left(\omega-l\frac{2\pi}{3}\right)k}\right)\right|\\
&= \left|F_0\left(e^{j3\left(\omega-l\frac{2\pi}{3}\right)}\right)\right|
\end{aligned}
$$

Therefore, we obtain $\left|F_l\left(e^{j\omega}\right)\right|$ by shifting $\left|F_0\left(e^{j\omega}\right)\right|$ to the right by $\frac{2\pi}{3}l$. Hence, the magnitude responses of $\left|F_0\left(e^{j\omega}\right)\right|$ and $\left|F_2\left(e^{j\omega}\right)\right|$ are as shown in Figure 4.

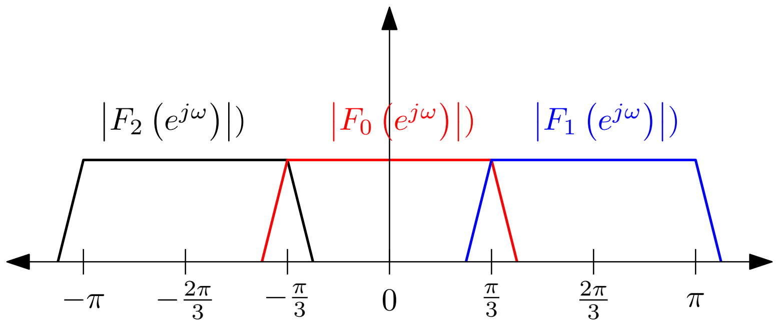

FIGURE 4. Magnitude responses of the synthesis filters $F_0\left(e^{j\omega}\right)$ and $F_2\left(e^{j\omega}\right)$ for the given $\left|F_1\left(e^{j\omega}\right)\right|$.

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Problem 6. P. P. Vaidyanathan, Problem #4.22. (15 pts)

Solution. We want to identify m_0 such that
1) $E_k(z)$ is an image of $E_{m_0-k}(z)$, $0 \le k \le m_0$
2) $E_k(z)$ is an image of $E_{M+m_0-k}(z)$, $m_0+1 \le k \le M-1$ i.e., $E_{m_0+k}(z)$ is an image of $E_{M-k}(z)$, $1 \le k \le M-1-m_0$

Part (a):

Consider a 7^{th} order filter that satisfies the constraint $h(n) = h(N-n), \quad 0 \le n \le N$.

$$H(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_3 z^{-4} + a_2 z^{-5} + a_1 z^{-6} + a_0 z^{-7}$$

For $M = 3$,

$$H(z) = \left(a_0 + a_3 z^{-3} + a_1 z^{-6}\right) + z^{-1}\left(a_1 + a_4 z^{-3} + a_0 z^{-6}\right) + z^{-2}\left(a_2 + a_2 z^{-3}\right)$$

$$\begin{aligned} E_0(z) &= a_0 + a_3 z^{-3} + a_1 z^{-6} \\ E_1(z) &= a_1 + a_4 z^{-3} + a_0 z^{-6} \\ E_2(z) &= a_2 + a_2 z^{-3}. \end{aligned}$$

In this case, $E_0(z)$ is an image of $E_1(z)$ and $E_2(z)$ is and image of itself. Therefore, $m_0 = 1$.
For $M = 4$,

$$H(z) = \left(a_0 + a_3 z^{-4}\right) + z^{-1}\left(a_1 + a_2 z^{-4}\right) + z^{-2}\left(a_2 + a_1 z^{-4}\right) + z^{-3}\left(a_3 + a_1 z^{-4}\right)$$

$$\begin{aligned} E_0(z) &= \left(a_0 + a_3 z^{-4}\right) \\ E_1(z) &= \left(a_1 + a_2 z^{-4}\right) \\ E_2(z) &= \left(a_2 + a_1 z^{-4}\right) \\ E_3(z) &= \left(a_3 + a_1 z^{-4}\right) \end{aligned}$$

In this case, $E_0(z)$ is an image of $E_3(z)$ and $E_1(z)$ is an image of $E_2(z)$. Therefore, $m_0 = 4$.

Part (b):

Let $h_0, h_1, \cdots, h_N$ be the coefficients of N^{th} order filter $H(z)$. Let $E_k(z)$ be the polyphase components of $H(z)$

$$H(z) = \sum_{i=0}^{M-1} z^{-i} E_i\left(z^M\right).$$

If $h_n = h_{N-n}$,

$$\begin{aligned} H(z) &= z^{-N} H\left(z^{-1}\right) \\ \Longrightarrow \sum_{i=0}^{M-1} z^{-i} E_i\left(z^M\right) &= z^{-N} \sum_{i=0}^{M-1} z^{i} E_i\left(z^{-M}\right) \\ &= z^{-N} \sum_{i=0}^{M-1} z^{i} E_i\left(z^{-M}\right) \quad j = M-1-i \\ &= z^{-N} \sum_{j=0}^{M-1} z^{M-1-j} E_{M-1-j}\left(z^{-M}\right) \\ &= z^{M-(N+1)} \sum_{j=0}^{M-1} z^{-j} E_{M-1-j}\left(z^{-M}\right) \end{aligned}$$

Let $N+1 = pM + q$ where $q = (N+1) \mod M$. We can write

$$\begin{aligned} \sum_{i=0}^{M-1} z^{-i} E_i\left(z^M\right) &= z^{M-pM-q} \sum_{j=0}^{M-1} z^{-j} E_{M-1-j}\left(z^{-M}\right) \\ &= \sum_{j=0}^{M-1} z^{-(j+q)} z^{(1-p)M} E_{M-1-j}\left(z^{-M}\right) \\ &= \sum_{j=0}^{M-1} z^{-(j+q)} z^{(1-p)M} E_{M-1-j}\left(z^{-M}\right) \end{aligned}$$

Case $q = 0$:

$$\begin{aligned}\sum_{i=0}^{M-1} z^{-i} E_i\left(z^M\right) &= \sum_{i=0}^{M-1} z^{-(i+q)} z^{(1-p)M} E_{M-1-i}\left(z^{-M}\right)\\ &= \sum_{i=0}^{M-1} z^{-i} \underbrace{z^{(1-p)M} E_{M-1-i}\left(z^{-M}\right)}_{\text{polynomial in } z^M}\end{aligned}$$

The R.H.S. and L.H.S. give the polyphase type-I decomposition in two different forms. Since type-I decomposition is unique, we equate the polyphase components from R.H.S and L.H.S. We get

$$E_i\left(z^M\right) = z^{(1-p)M} E_{M-1-i}\left(z^{-M}\right), \quad i = 1, 2, \cdots, M-1$$

Therefore, $E_{M-1-i}(z)$ is an image of $E_i\left(z^M\right)$. Therefore, $m_0 = M$.

Case $q > 0$:

$$\begin{aligned}\sum_{i=0}^{M-1} z^{-i} E_i\left(z^M\right) &= \sum_{i=0}^{M-1} z^{-(i+q)} z^{(1-p)M} E_{M-1-i}\left(z^{-M}\right)\\ &= \sum_{i=0}^{M-1-q} z^{-(i+q)} z^{(1-p)M} E_{M-1-i}\left(z^{-M}\right)\\ &\quad + \sum_{i=M-q}^{M-1} z^{-(i+q)} z^{(1-p)M} E_{M-1-i}\left(z^{-M}\right)\\ &= \sum_{j=q}^{M-1} z^{-j} z^{(1-p)M} E_{M-1-j+q}\left(z^{-M}\right) \quad j = i+q\\ &\quad + \sum_{j=0}^{q-1} z^{-(j+M)} z^{(1-p)M} E_{q-j-1}\left(z^{-M}\right) \quad j = i-(M-q)\\ \sum_{i=0}^{M-1} z^{-i} E_i\left(z^M\right) &= \sum_{j=q}^{M-1} z^{-j} \underbrace{z^{(1-p)M} E_{M-1-j+q}\left(z^{-M}\right)}_{\text{polynomial in } z^M}\\ &\quad + \sum_{j=0}^{q-1} z^{-j} \underbrace{z^{-pM} E_{q-j-1}\left(z^{-M}\right)}_{\text{polynomial in } z^M}\end{aligned}$$

The R.H.S. and L.H.S. give the polyphase type-I decomposition in two different forms. Since type-I decomposition is unique, we equate the polyphase components from R.H.S and L.H.S. We get

$$E_i\left(z^M\right) = \begin{cases} z^{-pM} E_{q-i-1}\left(z^{-M}\right), & 0 \le i < q \\ z^{(1-p)M} E_{M-1-i+q}\left(z^{-M}\right), & q \le i < M-1 \end{cases}$$

Therefore, $E_i(z)$ is an image of E_{q-1-i} for $0 \le i \le q-1$ and $E_i(z)$ is and image of $E_{M+q-1-i}$ for $q \le i < M-1$. Therefore, $m_0 = q-1$.

Therefore,

$$\begin{aligned} m_0 &= \begin{cases} M, & M \text{ divides } (N+1) \\ ((N+1) \mod M) - 1, & \text{otherwise} \end{cases}\\ m_0 &= \begin{cases} M, & M \text{ divides } (N+1) \\ N \mod M, & \text{otherwise} \end{cases}\end{aligned}$$

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Problem 7. P. P. Vaidyanathan, Problem #4.26. (10 pts)

Solution. Figure 5 shows the linear-phase interpolator.

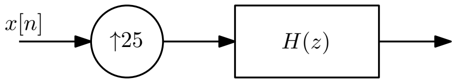

FIGURE 5. 25-fold low-pass linear-phase interpolator

(Part 1) Figure 6 shows the passband and stopband frequencies required for the interpolation filter. Hence the cutoff frequencies are given by

$$\omega_p = \frac{0.95\pi}{25}$$
$$\omega_s = \frac{2\pi - 0.95\pi}{25}$$

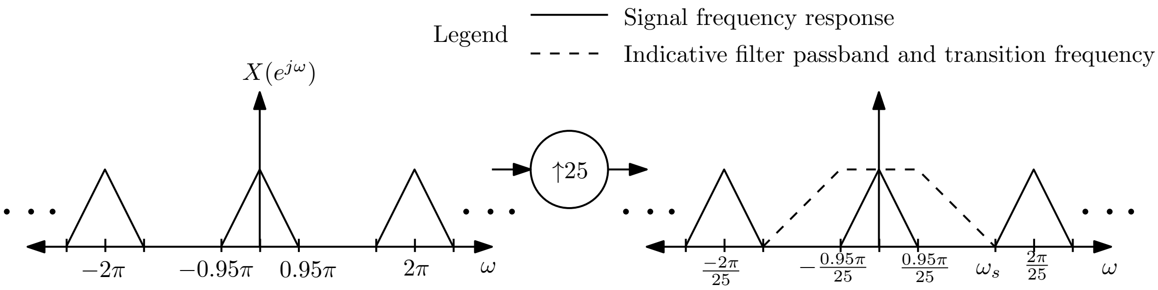

FIGURE 6. Upsampler in frequency domain

(Part 2) Filter order is given in Table 1.

(Part 3) Two stage implementation is shown in Figure 7. The frequency domain representation of the two stage interpolator is given in Figure 8.

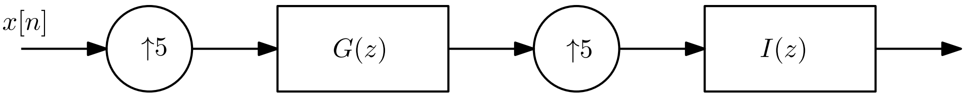

FIGURE 7. Two stage implementation of 25-fold low-pass linear phase interpolator

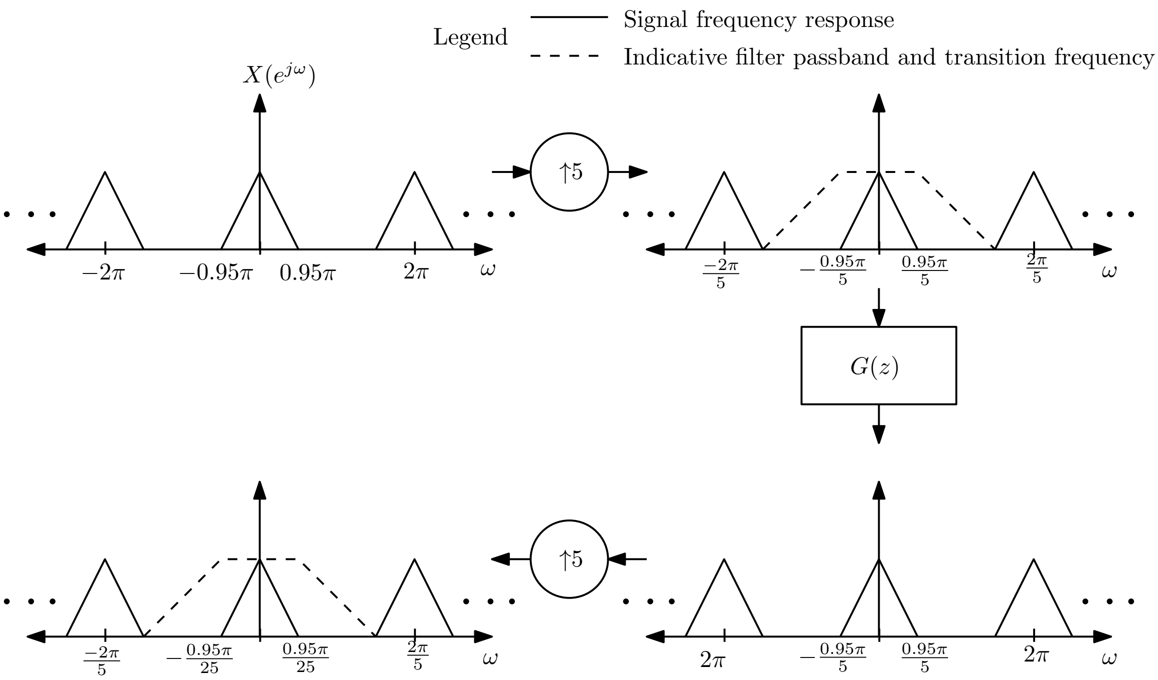

FIGURE 8. Frequency domain representation of 2-stage interpolator

We can choose the cutoff frequencies for $G(z)$ and $I(z)$ from Figure 8. For cascaded filters, passband ripples get added and stop band ripple remain same.

For $G(z)$, we have $\omega_p = \frac{0.95\pi}{5}$ and $\omega_s = \frac{2\pi - 0.95\pi}{5}$. Choose $\delta_p = \frac{\delta_1}{2} = 0.005$, $\delta_s = \delta_2 = 0.001$.

For $I(z)$, we have $\omega_p = \frac{0.95\pi}{25}$ and $\omega_s = \frac{2\pi}{5} - \frac{0.95\pi}{25}$. Choose $\delta_p = \frac{\delta_1}{2} = 0.005$, $\delta_s = \delta_2 = 0.001$.

The order of the filters is given in Table 1.

(Part 4) The sampling frequency is 8KHz. The computational complexity of the two implementations is given in Table 1.

Filter	$\frac{\Delta F}{F} = \frac{\omega_s - \omega_p}{2\pi}$	δ_p	δ_s	$D_\infty(\delta_p, \delta_s)$	Order(N)	Op Freq	(#muls per sec)
$H(z)$	0.002	0.01	0.002	13.9258	6963	200 kHz	1392600000
$G(z)$	0.01	0.005	0.001	15.6144	1562	40 kHz	62480000
$I(z)$	0.181	0.005	0.001	15.6144	86	200 kHz	17200000

TABLE 1. Interpolation filter parameters

Based on the computational complexity, we can say that the two stage implementation runs 17.4 $\left(= \frac{1392600000}{62480000+17200000}\right)$ times faster than the single stage implementation.

The filter order calculation is based on following empirical formula[1]

$$\begin{aligned} N &= \frac{D_\infty(\delta_p, \delta_s)}{\Delta F/F} \\ D_\infty(\delta_p, \delta_s) &= (\log_{10}\delta_s)\left[a_1(\log_{10}\delta_p)^2 + a_2\log_{10}\delta_p + a_3\right] \\ &\quad + \left[a_4(\log_{10}\delta_p)^2 + a_5\log_{10}\delta_p + a_6\right] \end{aligned}$$

where $a_1 = 5.3e-3, a_2 = 0.071, a_3 = -0.4761, a_4 = -0.0026, a_5 = -0.5941, a_6 = -0.4278$ ■

[1] O.Herman etal, *"Practical design rules for optimum low pass FIR digital filters"*, Bell-sys tech.Journal, vol 52,no.2,July 1973.

Problem 8. P. P. Vaidyanathan, Problem #5.2. (10 pts)

Solution. The modified QMF bank is shown in Figure 9.

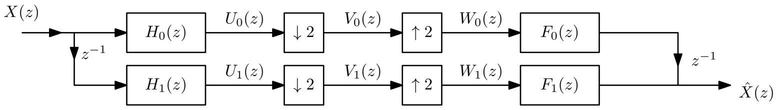

FIGURE 9. Modified 2 channel QMF bank.

$$
\begin{aligned}
U_0(z) &= H_0(z)X(z) \\
U_1(z) &= z^{-1}H_1(z)X(z) \\
V_0(z) &= \frac{1}{2}\left(U_0\left(z^{\frac{1}{2}}\right)+U_0\left(-z^{\frac{1}{2}}\right)\right) = \frac{1}{2}\left(H_0\left(z^{\frac{1}{2}}\right)X\left(z^{\frac{1}{2}}\right)+H_0\left(z^{\frac{1}{2}}\right)X\left(-z^{\frac{1}{2}}\right)\right) \\
V_1(z) &= \frac{1}{2}\left(U_1\left(z^{\frac{1}{2}}\right)+U_1\left(-z^{\frac{1}{2}}\right)\right) = \frac{1}{2}z^{-\frac{1}{2}}\left(H_1\left(z^{\frac{1}{2}}\right)X\left(z^{\frac{1}{2}}\right)-H_1\left(z^{\frac{1}{2}}\right)X\left(-z^{\frac{1}{2}}\right)\right) \\
W_0(z) &= V_0\left(z^2\right) = \frac{1}{2}\left(H_0(z)X(z)+H_0(-z)X(-z)\right) \\
W_1(z) &= V_1\left(z^2\right) = \frac{1}{2}z^{-1}\left(H_1(z)X(z)-H_1(-z)X(-z)\right) \\
\hat{X}(z) &= W_0(z)F_0(z)z^{-1}+W_1(z)F_1(z) \\
&= \frac{F_0(z)}{2}z^{-1}\left(H_0(z)X(z)+H_0(-z)X(-z)\right)+\frac{F_1(z)}{2}z^{-1}\left(H_1(z)X(z)-H_1(-z)X(-z)\right) \\
\hat{X}(z) &= z^{-1}\frac{X(z)}{2}\left(H_0(z)F_0(z)+H_1(z)F_1(z)\right) \\
&\quad +z^{-1}\frac{X(z)}{2}\left(H_0(-z)F_0(z)-H_1(-z)F_1(z)\right)
\end{aligned}
$$

If $H_0(z)=H_1(-z)$, $F_0(z)=H_0(z)$ and $F_1(z)=H_1(z)$,

$$
\begin{aligned}
\hat{X}(z) &= z^{-1}\frac{X(z)}{2}\left((H_0(z))^2+(H_0(-z))^2\right) \\
&\quad +z^{-1}\frac{X(z)}{2}\left(H_0(-z)H_0(z)+H_0(z)H_0(-z)\right) \\
\hat{X}(z) &= z^{-1}\frac{X(z)}{2}\left((H_0(z))^2+(H_0(-z))^2\right)
\end{aligned}
$$

Therefore, the choice of $F_0(z)$ and $F_1(z)$ cancels aliasing component $X(-z)$. The transfer function in this case is

$$
T(z)=\frac{\hat{X}(z)}{X(z)}=\frac{z^{-1}}{2}\left((H_0(z))^2+(H_0(-z))^2\right).
$$

If $H_0(z)$ is linear phase filter, we can write

$$
\begin{aligned}
H\left(e^{j\omega}\right) &= e^{-j\omega\frac{N}{2}}\left|H\left(e^{j\omega}\right)\right| \\
\Longrightarrow H\left(-e^{j\omega}\right) &= H\left(e^{j(\omega-\pi)}\right) = e^{-j\omega\frac{N}{2}}e^{+j\frac{N}{2}\pi}\left|H\left(e^{j(\omega-\pi)}\right)\right|.
\end{aligned}
$$

In this case, the transfer function is

$$
\begin{aligned}
T\left(e^{j\omega}\right) &= \frac{e^{-j\omega}}{2}\left(e^{-j\omega N}\left|H\left(e^{j\omega}\right)\right|^2+e^{-j\omega N}e^{jN\pi}\left|H\left(e^{j(\omega-\pi)}\right)\right|^2\right) \\
&= \frac{e^{-j\omega}e^{-j\omega N}}{2}\left(\left|H\left(e^{j\omega}\right)\right|^2+(-1)^N\left|H\left(e^{j(\omega-\pi)}\right)\right|^2\right)
\end{aligned}
$$

Therefore, system does not introduce phase distortion. Since, $\left|H\left(e^{j\omega}\right)\right|$ is even function in ω, $\left|H\left(e^{j\frac{\pi}{2}}\right)\right|^2=\left|H\left(e^{-j\frac{\pi}{2}}\right)\right|^2$. Therefore,

$$
\begin{aligned}
\left|T\left(e^{j\frac{\pi}{2}}\right)\right| &= \left|\left|H\left(e^{j\frac{\pi}{2}}\right)\right|^2+(-1)^N\left|H\left(e^{-j\frac{\pi}{2}}\right)\right|^2\right| \\
\left|T\left(e^{j\frac{\pi}{2}}\right)\right| &= \left|H\left(e^{-j\frac{\pi}{2}}\right)\right|^2\times\left|1+(-1)^N\right|
\end{aligned}
$$

Therefore, to avoid $T\left(e^{j\frac{\pi}{2}}\right)=0$, N has to be even.

Since N is a linear phase filter, the unique number of coefficients are $\frac{N}{2}+1$. Hence, the number of MPUs and APUs required is $\frac{N}{2}+1$. ■

Problem 9. P. P. Vaidyanathan, Problem #5.9. (15 pts)

Solution. The 2 channel QMF bank is shown in Figure 10.

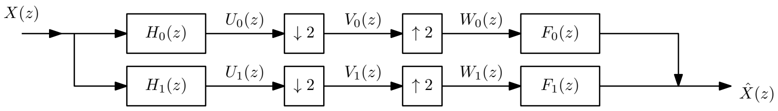

FIGURE 10. 2 channel QMF bank.

$$
\begin{aligned}
U_0(z) &= H_0(z)\,X(z) \\
U_1(z) &= H_1(z)\,X(z) \\
V_0(z) &= \frac{1}{2}\left(U_0\left(z^{\frac{1}{2}}\right) + U_0\left(-z^{\frac{1}{2}}\right)\right) = \frac{1}{2}\left(H_0\left(z^{\frac{1}{2}}\right)X\left(z^{\frac{1}{2}}\right) + H_0\left(z^{\frac{1}{2}}\right)X\left(-z^{\frac{1}{2}}\right)\right) \\
V_1(z) &= \frac{1}{2}\left(U_1\left(z^{\frac{1}{2}}\right) + U_1\left(-z^{\frac{1}{2}}\right)\right) = \frac{1}{2}\left(H_1\left(z^{\frac{1}{2}}\right)X\left(z^{\frac{1}{2}}\right) - H_1\left(z^{\frac{1}{2}}\right)X\left(-z^{\frac{1}{2}}\right)\right) \\
W_0(z) &= V_0\left(z^2\right) = \frac{1}{2}\left(H_0(z)\,X(z) + H_0(-z)\,X(-z)\right) \\
W_1(z) &= V_1\left(z^2\right) = \frac{1}{2}\left(H_1(z)\,X(z) - H_1(-z)\,X(-z)\right) \\
\hat{X}(z) &= W_0(z)\,F_0(z) + W_1(z)\,F_1(z) \\
&= \frac{F_0(z)}{2}\left(H_0(z)\,X(z) + H_0(-z)\,X(-z)\right) + \frac{F_1(z)}{2}\left(H_1(z)\,X(z) - H_1(-z)\,X(-z)\right) \\
\hat{X}(z) &= \frac{X(z)}{2}\left(H_0(z)\,F_0(z) + H_1(z)\,F_1(z)\right) \\
&\quad + \frac{X(z)}{2}\left(H_0(-z)\,F_0(z) - H_1(-z)\,F_1(z)\right)
\end{aligned}
$$

If $H_0(z) = H_1(-z)$,

$$
\begin{aligned}
\hat{X}(z) &= \frac{X(z)}{2}\left(H_0(z)\,F_0(z) + H_0(-z)\,F_1(z)\right) \\
&\quad + \frac{X(z)}{2}\left(H_0(-z)\,F_0(z) - H_0(z)\,F_1(z)\right)
\end{aligned}
$$

To avoid aliasing, we need

$$
\begin{aligned}
H_0(-z)\,F_0(z) - H_0(z)\,F_1(z) &= 0 \\
\Longrightarrow \frac{F_0(z)}{F_1(z)} &= \frac{H_0(z)}{H_0(-z)}
\end{aligned}
$$

Let

$$
\begin{aligned}
F_0(z) &= 2H_0(z)\,C(z) \\
F_1(z) &= 2H_0(-z)\,C(z)
\end{aligned}
$$

In this case, the transfer function is

$$
\begin{aligned}
T(z) = \frac{\hat{X}(z)}{X(z)} &= \left((H_0(z))^2 + (H_0(-z))^2\right)C(z) \\
&= \left[\left(E_0\left(z^2\right) + z^{-1}E_1\left(z^2\right)\right)^2 + \left(E_0\left(z^2\right) - z^{-1}E_1\left(z^2\right)\right)^2\right]C(z) \\
&= \left[\left(E_0\left(z^2\right)\right)^2 + z^{-2}\left(E_1\left(z^2\right)\right)^2\right]C(z) \\
\Longrightarrow C(z) &= \frac{T(z)}{\left(E_0\left(z^2\right)\right)^2 + z^{-2}\left(E_1\left(z^2\right)\right)^2}
\end{aligned}
$$

Therefore, the synthesis filters are

$$
\begin{aligned}
F_0(z) &= 2zH_0(z)\,\frac{T(z)}{\left(E_0\left(z^2\right)\right)^2 + z^{-2}\left(E_1\left(z^2\right)\right)^2} \\
F_1(z) &= 2zH_1(z)\,\frac{T(z)}{\left(E_0\left(z^2\right)\right)^2 + z^{-2}\left(E_1\left(z^2\right)\right)^2}
\end{aligned}
$$

Since, $H_0(z)$ and $H_1(z)$ are stable, we need to identify $T(z)$ satisfying following conditions:

a) $T(z)$ is all pass.

b) $\frac{T(z)}{(E_0(z^2))^2+z^{-2}(E_1(z^2))^2}$ has poles inside unit circle i.e., if $\left(E_0\left(z^2\right)\right)^2 + z^{-2}\left(E_1\left(z^2\right)\right)^2$ has zeros outsize unit circle, $T(z)$ should also have zeros at the same locations.

If $a_1, a_2, \cdots, a_N$ are zeros of $\left(E_0\left(z^2\right)\right)^2 + z^{-2}\left(E_1\left(z^2\right)\right)^2$ that are located outside unit circle, we will choose $T(z)$ such that it satisfies above conditions:

$$T(z) = \prod_{i=1}^{N}\left(\frac{1-a_i z^{-1}}{a_i - z^{-1}}\right)$$

The stable synthesis filters that achieve this transfer function are

$$\begin{aligned} F_0(z) &= \frac{2H_0(z)}{\left(E_0\left(z^2\right)\right)^2 + z^{-2}\left(E_1\left(z^2\right)\right)^2}\prod_{i=1}^{N}\left(\frac{1-a_i z^{-1}}{a_i - z^{-1}}\right), \\ F_1(z) &= \frac{2H_0(-z)}{\left(E_0\left(z^2\right)\right)^2 + z^{-2}\left(E_1\left(z^2\right)\right)^2}\prod_{i=1}^{N}\left(\frac{1-a_i z^{-1}}{a_i - z^{-1}}\right). \end{aligned}$$

■

Problem 10. P. P. Vaidyanathan, Problem #5.2. (15 pts)

Solution. The 3 channel QMF bank is shown in Figure 11.

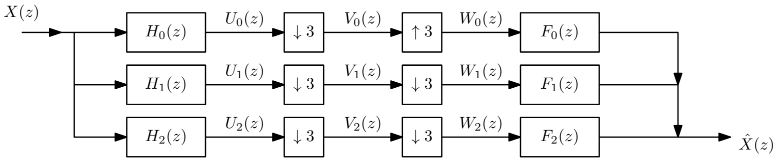

FIGURE 11. 3 channel QMF bank.

$$
\begin{aligned}
U_k(z) &= H_0(z)\, X(z), \quad k = 0, 1, 2. \\
V_k(z) &= \frac{1}{3}\sum_{i=0}^{2} U_k\left((\omega_3)^i z^{\frac{1}{3}}\right), \quad \omega_3 = e^{j\frac{2\pi}{3}} \\
&= \frac{1}{3}\sum_{i=0}^{2} H_k\left((\omega_3)^i z^{\frac{1}{3}}\right) X\left((\omega_3)^i z^{\frac{1}{3}}\right) \\
W_k(z) &= U_k\left(z^3\right) \\
&= \frac{1}{3}\sum_{i=0}^{2} H_k\left((\omega_3)^i z\right) X\left((\omega_3)^i z\right) \\
\hat{X}(z) &= \frac{1}{3}\sum_{k=0}^{2} W_k(z)\, F_k(z) \\
&= \frac{1}{3}\sum_{k=0}^{2}\left(F_k(z)\sum_{i=0}^{2} H_k\left((\omega_3)^i z\right) X\left((\omega_3)^i z\right)\right) \\
\hat{X}(z) &= \sum_{i=0}^{2}\left[X\left((\omega_3)^i z\right)\left(\sum_{k=0}^{2} F_k(z)\, H_k\left((\omega_3)^i z\right)\right)\right]
\end{aligned}
$$

To avoid aliasing, we require

$$
\begin{aligned}
\sum_{k=0}^{2} F_k(z)\, H_k(z) &= 3T(z) \neq 0 \\
\sum_{k=0}^{2} F_k(z)\, H_k(\omega_3 z) &= 0 \\
\sum_{k=0}^{2} F_k(z)\, H_k\left(\omega_3^2 z\right) &= 0,
\end{aligned}
$$

where $T(z) = \frac{\hat{X}(z)}{X(z)}$. These conditions can be written in matrix form as

$$
\begin{bmatrix} H_0(z) & H_1(z) & H_2(z) \\ H_0(\omega_3 z) & H_1(\omega_3 z) & H_2(\omega_3 z) \\ H_0\left(\omega_3^2 z\right) & H_1\left(\omega_3^2 z\right) & H_2\left(\omega_3^2 z\right) \end{bmatrix}
\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix}
=
\begin{bmatrix} 3T(z) \\ 0 \\ 0 \end{bmatrix}
$$

For perfect reconstruction, we require $T(z) = z^{-k}$ for some integer k.

$$
\begin{bmatrix} H_0(z) & H_1(z) & H_2(z) \\ H_0(\omega_3 z) & H_1(\omega_3 z) & H_2(\omega_3 z) \\ H_0\left(\omega_3^2 z\right) & H_1\left(\omega_3^2 z\right) & H_2\left(\omega_3^2 z\right) \end{bmatrix}
\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix}
= 3z^{-k}
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

For the set of analysis filters, $H_0(z) = 1$.

$$
\begin{bmatrix} 1 & H_1(z) & H_2(z) \\ 1 & H_1(\omega_3 z) & H_2(\omega_3 z) \\ 1 & H_1\left(\omega_3^2 z\right) & H_2\left(\omega_3^2 z\right) \end{bmatrix}
\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix}
= 3z^{-k}
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

We have $\sum_{i=0}^{2} H_1\left(\omega_3^i z\right) = 3h_1(0) = 6$ and $\sum_{i=0}^{2} H_2\left(\omega_3^i z\right) = 3h_2(0) = 9$. By adding 2^{nd} and 3^{rd} row to the 1^{st} row, we get

$$\begin{aligned}
\begin{bmatrix} 3 & 6 & 9 \\ 1 & H_1(\omega_3 z) & H_2(\omega_3 z) \\ 1 & H_1(\omega_3^2 z) & H_2(\omega_3^2 z) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} &= 3z^{-k} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\implies \begin{bmatrix} 1 & 2 & 3 \\ 1 & H_1(\omega_3 z) & H_2(\omega_3 z) \\ 1 & H_1(\omega_3^2 z) & H_2(\omega_3^2 z) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} &= 3z^{-k} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\implies F_0(z) + 2F_1(z) + 3F_2(z) &= 3z^{-k}
\end{aligned}$$

$$\implies \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = 3z^{-k} \begin{bmatrix} 1 & 2 & 3 \\ 1 & H_1(\omega_3 z) & H_2(\omega_3 z) \\ 1 & H_1(\omega_3^2 z) & H_2(\omega_3^2 z) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$\mathbf{H} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & H_1(\omega_3 z) & H_2(\omega_3 z) \\ 1 & H_1(\omega_3^2 z) & H_2(\omega_3^2 z) \end{bmatrix}.$$

Therefore, the first column of the $\mathbf{H}^{-1}$ gives the synthesis filters.
Let

$$\begin{aligned}
D(z) &= \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 1 & H_1(\omega_3 z) & H_2(\omega_3 z) \\ 1 & H_1(\omega_3^2 z) & H_2(\omega_3^2 z) \end{bmatrix} \right) \\
&= H_1(\omega_3 z) H_2(\omega_3^2 z) - H_2(\omega_3 z) H_1(\omega_3^2 z) \\
&\quad +2\left(H_2(\omega_3 z) - H_2(\omega_3^2 z)\right) \\
&\quad +3\left(H_1(\omega_3^2 z) - H_1(\omega_3 z)\right)
\end{aligned}$$

Therefore, the synthesis filters are

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \frac{3z^{-k}}{D(z)} \begin{bmatrix} H_1(\omega_3 z) H_2(\omega_3^2 z) - H_2(\omega_3 z) H_1(\omega_3^2 z) \\ H_2(\omega_3 z) - H_2(\omega_3^2 z) \\ H_1(\omega_3^2 z) - H_1(\omega_3 z) \end{bmatrix}$$

a) $H_0(z) = 1$, $H_1(z) = 2 + z^{-1}$, $H_2 = 3 + 2z^{-1} + z^{-2}$

$$\begin{aligned}
H_1(\omega_3 z) H_2(\omega_3^2 z) &= \left(2 + \omega_3^2 z^{-1}\right)\left(3 + 2\omega_3 z^{-1} + \omega_3^2 z^{-2}\right) \\
&= 6 + \left(4\omega_3 + 3\omega_3^2\right) z^{-1} + \left(2 + 2\omega_3^2\right) z^{-2} + \omega_3 z^{-3} \\
H_1(\omega_3^2 z) H_2(\omega_3 z) &= 6 + \left(4\omega_3^2 + 3\omega_3\right) z^{-1} + \left(2 + 2\omega_3\right) z^{-2} + \omega_3^2 z^{-3}
\end{aligned}$$

$H_1(\omega_3 z) H_2(\omega_3^2 z) - H_1(\omega_3^2 z) H_2(\omega_3 z) = -\sqrt{3} j z^{-1}\left(1 - 2z^{-1} + z^{-2}\right)$
$H_2(\omega_3 z) - H_2(\omega_3^2 z) = -\sqrt{3} j z^{-1}\left(-2 + z^{-1}\right)$
$H_1(\omega_3^2 z) - H_1(\omega_3 z) = -\sqrt{3} j z^{-1}$
$D(z) = -\sqrt{3} j z^{-3}$

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = 3z^{2-k} \begin{bmatrix} 1 - 2z^{-1} + z^{-2} \\ -2 + z^{-1} \\ 1 \end{bmatrix}$$

The synthesis filters are FIR filters.
b) $H_0(z) = 1$, $H_1(z) = 2 + z^{-1} + z^{-5}$, $H_2 = 3 + 2z^{-1} + z^{-2}$

$$\begin{aligned}
H_1(\omega_3 z) H_2(\omega_3^2 z) &= \left(2 + \omega_3^2 z^{-1} + \omega_3 z^{-5}\right)\left(3 + 2\omega_3 z^{-1} + \omega_3^2 z^{-2}\right) \\
&= 6 + \left(4\omega_3 + 3\omega_3^2\right) z^{-1} + \left(2 + 2\omega_3^2\right) z^{-2} + \omega_3 z^{-3} \\
&\quad +3\omega_3 z^{-5} + 2\omega_3^2 z^{-6} + z^{-7} \\
H_1(\omega_3^2 z) H_2(\omega_3 z) &= 6 + \left(4\omega_3^2 + 3\omega_3\right) z^{-1} + \left(2 + 2\omega_3\right) z^{-2} + \omega_3^2 z^{-3} \\
&\quad +3\omega_3^2 z^{-5} + 2\omega_3 z^{-6} + z^{-7}
\end{aligned}$$

$H_1(\omega_3 z) H_2(\omega_3^2 z) - H_1(\omega_3^2 z) H_2(\omega_3 z) = -\sqrt{3} j z^{-1}\left(1 - 2z^{-1} + z^{-2} + 3z^{-4} - 2z^{-5}\right)$
$H_2(\omega_3 z) - H_2(\omega_3^2 z) = -\sqrt{3} j z^{-1}\left(-2 + z^{-1}\right)$
$H_1(\omega_3^2 z) - H_1(\omega_3 z) = -\sqrt{3} j z^{-1}\left(1 - z^{-4}\right)$
$D(z) = -\sqrt{3} j z^{-1}\left(z^{-2} - 2z^{-5}\right)$

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \frac{3z^{2-k}}{1-2z^{-3}} \begin{bmatrix} 1-2z^{-1}+z^{-2}+3z^{-4}-2z^{-5} \\ -2+z^{-1} \\ 1-z^{-3} \end{bmatrix}$$

The synthesis filters are IIR filters with poles at $2^{\frac{1}{3}}, \omega_3 2^{\frac{1}{3}}$ and $\omega_3^2 2^{\frac{1}{3}}$. Since the poles lie outside the unit circle, the filters are unstable.

b) $H_0(z) = 1$, $H_1(z) = 2 + z^{-1} + z^{-5}$, $H_2 = 3 + z^{-1} + 2z^{-2}$

$$\begin{aligned} H_1(\omega_3 z) H_2(\omega_3^2 z) &= \left(2+\omega_3^2 z^{-1} + \omega_3 z^{-5}\right)\left(3+\omega_3 z^{-1} + 2\omega_3^2 z^{-2}\right) \\ &= 6 + \left(2\omega_3 + 3\omega_3^2\right) z^{-1} + \left(1+4\omega_3^2\right) z^{-2} + 2\omega_3 z^{-3} \\ &\quad +3\omega_3 z^{-5} + 1\omega_3^2 z^{-6} + 2z^{-7} \\ H_1(\omega_3^2 z) H_2(\omega_3 z) &= 6 + \left(2\omega_3 + 3\omega_3^2\right) z^{-1} + \left(1+4\omega_3^2\right) z^{-2} + 2\omega_3 z^{-3} \\ &\quad +3\omega_3 z^{-5} + 1\omega_3^2 z^{-6} + 2z^{-7} \end{aligned}$$

$H_1(\omega_3 z) H_2(\omega_3^2 z) - H_1(\omega_3^2 z) H_2(\omega_3 z) = -\sqrt{3} j z^{-1}\left(-1-4z^{-1}+2z^{-2}+3z^{-4}-z^{-5}\right)$

$H_2(\omega_3 z) - H_2(\omega_3^2 z) = -\sqrt{3} j z^{-1}\left(-1+2z^{-1}\right)$

$H_1(\omega_3^2 z) - H_1(\omega_3 z) = -\sqrt{3} j z^{-1}\left(1-z^{-4}\right)$

$D(z) = -\sqrt{3} j z^{-1}\left(2z^{-2} - 2z^{-5}\right)$

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \frac{3z^{2-k}}{2\left(1-z^{-3}\right)} \begin{bmatrix} \left(-1-4z^{-1}+2z^{-2}+3z^{-4}-z^{-5}\right) \\ -1+2z^{-1} \\ 1-z^{-4} \end{bmatrix}$$

The synthesis filters are IIR filters with poles at $1, \omega_3$ and ω_3^2. Since the poles lie on the unit circle, the filters are unstable. ■