Quantum Information Theory (E2-270) (Spring 2025) **HW4**
Instructor: Prof. Shayan Srinivasa Garani

1. **PROBLEM 1:** Compute the mutual information for the following: (a) quantum dephasing channel with parameter p and (b) quantum erasure channel with parameter ϵ

   Solution:

   The quantum mutual information of a quantum channel $\mathcal{N}$, acting on one half of a maximally entangled state $|\Phi^+\rangle$, is given by:

$$I(\mathcal{N}) = S(\rho_A) + S(\mathcal{N}(\rho_A)) - S((\mathcal{I} \otimes \mathcal{N})(|\Phi^+\rangle\langle\Phi^+|))$$

   where:

   - $S(\rho)$ is the von Neumann entropy,
   - $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is the maximally entangled Bell state,
   - ρ_A is the reduced density matrix on the first subsystem.

   (a) Quantum Dephasing Channel: The dephasing channel is defined as:

$$\mathcal{N}_{\text{deph}}(\rho) = (1-p)\rho + pZ\rho Z$$

   where Z is the Pauli-Z operator. Applying the channel to half of $|\Phi^+\rangle$, we get:

$$(\mathcal{I} \otimes \mathcal{N}_{\text{deph}})(|\Phi^+\rangle\langle\Phi^+|) = (1-p)|\Phi^+\rangle\langle\Phi^+| + p|\Phi^-\rangle\langle\Phi^-|$$

   where $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. This is a classical mixture of two orthogonal Bell states with probabilities $1-p$ and p, so its entropy is:

$$S = H(p) = -p\log_2 p - (1-p)\log_2(1-p)$$

   Since $\rho_A = \frac{I}{2}$, we have $S(\rho_A) = 1$. Similarly, $\mathcal{N}(\rho_A) = \frac{I}{2} \Rightarrow S(\mathcal{N}(\rho_A)) = 1$. Therefore, the mutual information is:

$$I(\mathcal{N}_{\text{deph}}) = 1 + 1 - H(p) = 2 - H(p)$$

   (b)Quantum erasure channel The quantum erasure channel $\mathcal{N}_{\text{erase}}$ with erasure probability ϵ is defined as:

$$\mathcal{N}_{\text{erase}}(\rho) = (1-\epsilon)\rho + \epsilon|e\rangle\langle e|,$$

   where $|e\rangle$ is orthogonal to the input space. The mutual information $I(X:Y)$ for an input ensemble $\{p_x, \rho_x\}$ is:

$$I(X:Y) = S\left(\mathcal{N}_{\text{erase}}(\rho)\right) - \sum_x p_x S\left(\mathcal{N}_{\text{erase}}(\rho_x)\right).$$

For the optimal ensemble of orthogonal pure states with $\rho = I/d$, the output entropy is:

$$S\left(\mathcal{N}_{\text{erase}}(\rho)\right) = H_2(\epsilon) + (1-\epsilon)\log d,$$

where $H_2(\epsilon) = -\epsilon\log\epsilon - (1-\epsilon)\log(1-\epsilon)$. The conditional entropy for pure states is:

$$S\left(\mathcal{N}_{\text{erase}}(|\psi_x\rangle\langle\psi_x|)\right) = H_2(\epsilon).$$

Thus, the mutual information simplifies to:

$$I(X:Y) = (1-\epsilon)\log d.$$

The classical capacity of the channel is:

$$C(\mathcal{N}_{\text{erase}}) = (1-\epsilon)\log d.$$

For a qubit ($d=2$):

$$C(\mathcal{N}_{\text{erase}}) = 1-\epsilon.$$

2. **PROBLEM 2:** Prove that the coherent information $Q(\mathcal{N}_1 \otimes \mathcal{N}_2)$ of a tensor product channel $\mathcal{N}_1 \otimes \mathcal{N}_2$ is never less than the sum of coherent informations over the individual channels, i.e.,

$$Q(\mathcal{N}_1 \otimes \mathcal{N}_2) \geq Q(\mathcal{N}_1) + Q(\mathcal{N}_2).$$

Solution: The coherent information of a quantum channel $\mathcal{N}$ for an input state ρ is defined as:

$$I_c(\rho, \mathcal{N}) = S(\mathcal{N}(\rho)) - S((\mathcal{I}\otimes\mathcal{N})(|\psi\rangle\langle\psi|))$$

where $|\psi\rangle$ is a purification of ρ, and $\mathcal{I}$ is the identity channel acting on the purifying system.

The coherent information of the channel is then:

$$Q(\mathcal{N}) = \max_{\rho} I_c(\rho, \mathcal{N})$$

Let ρ_1 and ρ_2 be the optimal input states achieving the coherent informations $Q(\mathcal{N}_1)$ and $Q(\mathcal{N}_2)$, respectively. Then consider the product input state $\rho_{12} = \rho_1 \otimes \rho_2$ for the product channel $\mathcal{N}_1 \otimes \mathcal{N}_2$.

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be purifications of ρ_1 and ρ_2, respectively, so that $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ purifies ρ_{12}.

Then, the coherent information of the product channel for this input is:

$$\begin{aligned}
I_c(\rho_{12}, \mathcal{N}_1\otimes\mathcal{N}_2) &= S\left((\mathcal{N}_1\otimes\mathcal{N}_2)(\rho_{12})\right) - S\left((\mathcal{I}\otimes\mathcal{N}_1\otimes\mathcal{N}_2)(|\psi_{12}\rangle\langle\psi_{12}|)\right) \\
&= S(\mathcal{N}_1(\rho_1)) + S(\mathcal{N}_2(\rho_2)) - [S((\mathcal{I}\otimes\mathcal{N}_1)(|\psi_1\rangle\langle\psi_1|)) \\
&\quad + S((\mathcal{I}\otimes\mathcal{N}_2)(|\psi_2\rangle\langle\psi_2|))] \\
&= I_c(\rho_1, \mathcal{N}_1) + I_c(\rho_2, \mathcal{N}_2)
\end{aligned}$$

Therefore,

$$Q(\mathcal{N}_1 \otimes \mathcal{N}_2) \geq I_c(\rho_{12}, \mathcal{N}_1 \otimes \mathcal{N}_2) = Q(\mathcal{N}_1) + Q(\mathcal{N}_2)$$

Conclusion: The coherent information is *superadditive*, meaning:

$$Q(\mathcal{N}_1 \otimes \mathcal{N}_2) \geq Q(\mathcal{N}_1) + Q(\mathcal{N}_2)$$

as required.

Acknowledgement: The solutions are prepared by Sudhir Kumar Sahoo and Abhi Kumar Sharma.