Quantum Information Theory (E2-270) (Spring 2025) **HW3**
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1. **PROBLEM 1: Exercise 5.2.9 (Mark Wilde)** Show that the matrix representation of an isometric extension $U_{A\to BE}$ of the amplitude damping channel is

$$\begin{bmatrix} \langle 0|_B \langle 0|_E U_{A\to BE} |0\rangle_A & \langle 0|_B \langle 0|_E U_{A\to BE} |1\rangle_A \\ \langle 0|_B \langle 1|_E U_{A\to BE} |0\rangle_A & \langle 0|_B \langle 1|_E U_{A\to BE} |1\rangle_A \\ \langle 1|_B \langle 0|_E U_{A\to BE} |0\rangle_A & \langle 1|_B \langle 0|_E U_{A\to BE} |1\rangle_A \\ \langle 1|_B \langle 1|_E U_{A\to BE} |0\rangle_A & \langle 1|_B \langle 1|_E U_{A\to BE} |1\rangle_A \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{1-\gamma} \\ 1 & 0 \\ 0 & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix},$$

where γ (with $0 \le \gamma \le 1$) is the damping parameter of the amplitude damping channel.
Solution: We need to demonstrate that the given matrix represents the isometric extension $U_{A\to BE}$ of the amplitude damping channel.

The amplitude damping channel models a quantum system losing energy to an environment, typically a qubit decaying from the excited state $|1\rangle$ to the ground state $|0\rangle$ with a probability related to γ. The Kraus operators for this channel are:

$$N_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}\,|1\rangle\langle 1|, \quad N_1 = \sqrt{\gamma}\,|0\rangle\langle 1|.$$

The isometric extension of the amplitude damping channel $\mathcal{N}_{A\to B}$ is

$$U^{\mathcal{N}}_{A\to BE} = \sum_j N_j \otimes |j\rangle_E$$

where N_j's are the Kraus operators of the channel. For an amplitude damping channel,

$$N_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix} \Rightarrow |0\rangle\langle 0| + \sqrt{\gamma}\,|1\rangle\langle 1|$$

and

$$N_1 = \begin{pmatrix} 0 & \sqrt{1-\gamma} \\ 0 & 0 \end{pmatrix} \Rightarrow |0\rangle\langle 1|\sqrt{1-\gamma}$$

Let's compute elements of isometric extension of $U^{\mathcal{N}}_{A\to BE}$:

$$\left(U^{\mathcal{N}}_{A\to BE}\right)_{00} = \langle 0|_B \langle 0|_E U^{\mathcal{N}}_{A\to BE} |0\rangle_A$$

$$= \langle 0|_B \langle 0|_E \left(\sum_j N_j \otimes |j\rangle_E\right) |0\rangle_A$$

$$= \sum_j \langle 0_B|N_j|0_A\rangle \langle 0_E|j_E\rangle$$

$$= \langle 0_B|N_0|0_A\rangle \langle 0_E|0_E\rangle + \langle 0_B|N_1|0_A\rangle \langle 0_E|1_E\rangle$$

$$\Rightarrow \left(U^{\mathcal{N}}_{A\to BE}\right)_{00} = 0$$

Now,

$$\left(U^{\mathcal{N}}_{A\to BE}\right)_{01} = \langle 0|_B \langle 0|_E \, U^{\mathcal{N}}_{A\to BE} \, |1\rangle_A$$

$$= \sum_j \langle 0_B|N_j|1_A\rangle \langle 0_E|j_E\rangle$$

$$= \langle 0_B|N_0|1_A\rangle \langle 0_E|0_E\rangle + \langle 0_B|N_1|1_A\rangle \langle 0_E|1_E\rangle$$

$$\Rightarrow \left(U^{\mathcal{N}}_{A\to BE}\right)_{01} = \sqrt{\gamma}$$

Again,

$$\left(U^{\mathcal{N}}_{A\to BE}\right)_{10} = \langle 0|_B \langle 1|_E \; U^{\mathcal{N}}_{A\to BE} \, |0\rangle_A$$

$$= \sum_j \langle 0|_B \, N_j \, |0\rangle_A \, \langle 1|j\rangle_E$$

$$= \langle 0_B|N_0|0_A\rangle \langle 1_E|0_E\rangle + \langle 0_B|N_1|0_A\rangle \langle 1_E|1_E\rangle = 1$$

Also,

$$\left(U^{\mathcal{N}}_{A\to BE}\right)_{11} = \langle 0|_B \langle 1|_E \, U^{\mathcal{N}}_{A\to BE} \, |1\rangle_A$$

$$= \sum_j \langle 0_B|N_j|1_A\rangle \langle 1_E|j_E\rangle$$

$$= \langle 0_B|N_0|1_A\rangle \langle 1_E|0_E\rangle + \langle 0_B|N_1|1_A\rangle \langle 1_E|1_E\rangle$$

$$\Rightarrow \left(U^{\mathcal{N}}_{A\to BE}\right)_{11} = 0$$

Consider,

$$\left(U^{\mathcal{N}}_{A\to BE}\right)_{20} = \langle 1|_B \langle 0|_E \, U^{\mathcal{N}}_{A\to BE} \, |0\rangle_A$$

$$= \sum_j \langle 1_B|N_j|0_A\rangle \langle 0_E|j_E\rangle$$

$$= \langle 1_B|N_0|0_A\rangle \langle 0_E|0_E\rangle + \langle 1_B|N_1|0_A\rangle \langle 0_E|1_E\rangle$$

$$= 0 \cdot 1 + 0 \cdot 0 = 0$$

$$\Rightarrow \left(U^{\mathcal{N}}_{A\to BE}\right)_{20} = 0$$

Consider,

$$\left(U^{\mathcal{N}}_{A\to BE}\right)_{21} = \langle 1|_B \langle 0|_E \, U^{\mathcal{N}}_{A\to BE} \, |1\rangle_A$$

$$= \sum_j \langle 1_B|N_j|1_A\rangle \langle 0_E|j_E\rangle$$

$$= \langle 1_B|N_0|1_A\rangle \langle 0_E|0_E\rangle + \langle 1_B|N_1|1_A\rangle \langle 0_E|1_E\rangle = 0$$

$$\Rightarrow \left(U^{\mathcal{N}}_{A\to BE}\right)_{21} = 0$$

Consider,

$$\left(U^{\mathcal{N}}_{A\to BE}\right)_{30} = \langle 1|_B \langle 1|_E \, U^{\mathcal{N}}_{A\to BE} \, |0\rangle_A$$

$$= \sum_j \langle 1_B|N_j|0_A\rangle \langle 1_E|j_E\rangle$$

$$= \langle 1_B|N_0|0_A\rangle \langle 1_E|0_E\rangle + \langle 1_B|N_1|0_A\rangle \langle 1_E|1_E\rangle$$

$$= 0 \cdot 0 + 0 \cdot 1 = 0$$

$$\Rightarrow \left(U^{\mathcal{N}}_{A\to BE}\right)_{30} = 0$$

Consider,

$$\left(U^{\mathcal{N}}_{A\to BE}\right)_{31} = \langle 1|_B \langle 1|_E U^{\mathcal{N}}_{A\to BE} |1\rangle_A$$

$$= \sum_j \langle 1_B|N_j|1_A\rangle \langle 1_E|j_E\rangle$$

$$= \langle 1_B|N_0|1_A\rangle \langle 1_E|0_E\rangle + \langle 1_B|N_1|1_A\rangle \langle 1_E|1_E\rangle$$

$$= \sqrt{\gamma} \cdot 0 + \sqrt{1-\gamma} \cdot 1 = \sqrt{1-\gamma}$$

$$\Rightarrow \left(U^{\mathcal{N}}_{A\to BE}\right)_{31} = \sqrt{1-\gamma}$$

Thus, the solution is:

$$\boxed{\begin{bmatrix} \langle 0|_B \langle 0|_E U_{A\to BE} |0\rangle_A & \langle 0|_B \langle 0|_E U_{A\to BE} |1\rangle_A \\ \langle 0|_B \langle 1|_E U_{A\to BE} |0\rangle_A & \langle 0|_B \langle 1|_E U_{A\to BE} |1\rangle_A \\ \langle 1|_B \langle 0|_E U_{A\to BE} |0\rangle_A & \langle 1|_B \langle 0|_E U_{A\to BE} |1\rangle_A \\ \langle 1|_B \langle 1|_E U_{A\to BE} |0\rangle_A & \langle 1|_B \langle 1|_E U_{A\to BE} |1\rangle_A \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{1-\gamma} \\ 1 & 0 \\ 0 & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix}}$$

Exercise 5.2.10 (Mark Wilde) Consider a full unitary $V_{AE\to BE}$ such that

$$\mathrm{Tr}_E\{V(\rho_A \otimes |0\rangle \langle 0|_E)V^\dagger\}$$

gives the amplitude damping channel. Show that a matrix representation of V is

$$\begin{bmatrix} \langle 0|_B \langle 0|_E V |0\rangle_A \langle 0|_E V |0\rangle_A & \langle 0|_B \langle 0|_E V |0\rangle_A \langle 0|_E V |1\rangle_A & \langle 0|_B \langle 0|_E V |1\rangle_A \langle 0|_E V |0\rangle_A & \langle 0|_B \langle 0|_E V |1\rangle_A \langle 0|_E V |1\rangle_A \\ \langle 0|_B \langle 1|_E V |0\rangle_A \langle 0|_E V |0\rangle_A & \langle 0|_B \langle 1|_E V |0\rangle_A \langle 0|_E V |1\rangle_A & \langle 0|_B \langle 1|_E V |1\rangle_A \langle 0|_E V |0\rangle_A & \langle 0|_B \langle 1|_E V |1\rangle_A \langle 0|_E V |1\rangle_A \\ \langle 1|_B \langle 0|_E V |0\rangle_A \langle 0|_E V |0\rangle_A & \langle 1|_B \langle 0|_E V |0\rangle_A \langle 0|_E V |1\rangle_A & \langle 1|_B \langle 0|_E V |1\rangle_A \langle 0|_E V |0\rangle_A & \langle 1|_B \langle 0|_E V |1\rangle_A \langle 0|_E V |1\rangle_A \\ \langle 1|_B \langle 1|_E V |0\rangle_A \langle 0|_E V |0\rangle_A & \langle 1|_B \langle 1|_E V |0\rangle_A \langle 0|_E V |1\rangle_A & \langle 1|_B \langle 1|_E V |1\rangle_A \langle 0|_E V |0\rangle_A & \langle 1|_B \langle 1|_E V |1\rangle_A \langle 0|_E V |1\rangle_A \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & 0 & \sqrt{\gamma} & 0 \end{bmatrix}.$$

Solution:

We need to show that the given matrix represents the isometric extension V of the amplitude damping channel. Understanding the Problem Here, V is a unitary operator acting on A and E, mapping to B and E, such that tracing out E from $V(\rho_A \otimes |0\rangle \langle 0|_E)V^\dagger$ yields the amplitude damping channel. The matrix is 4×4, representing outer products of the action of V on $|0\rangle_A |0\rangle_E$ and $|1\rangle_A |0\rangle_E$, projected onto B and E. The basis for BE is $\{|00\rangle_{BE}, |10\rangle_{BE}, |01\rangle_{BE}, |11\rangle_{BE}\}$, and each element is a product of inner products.

We know $U^N_{AE\to BE}$, hence we can calculate the action V on the basis $|00\rangle_{AE}$ and $|10\rangle_{AE}$ as follows:

$$V|00\rangle_{AE} = U^N_{A\to BE}|0\rangle = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 1 & 0 \\ 0 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle_{AE}$$

Similarly,

$$V|10\rangle_{AE} = U^N_{A\to BE}|1\rangle = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 1 & 0 \\ 0 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{\gamma} \\ 0 \\ 0 \\ \sqrt{1-\gamma} \end{bmatrix} = \sqrt{\gamma}|00\rangle_{AE} + \sqrt{1-\gamma}|11\rangle_{AE}$$

Now to evaluate the action of V on the remaining two basis states $|01\rangle_{AE}$ and $|11\rangle_{AE}$. Let us assume

$$V|01\rangle_{AC} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad V|11\rangle_{AC} = \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}$$

We know that V is unitary, hence norm will be 1, which means

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$$

and

$$|e|^2 + |f|^2 + |g|^2 + |h|^2 = 1$$

Further,

$$\langle 11|01\rangle = 0 \Rightarrow \langle 11| V^\dagger V |01\rangle = 0 \Rightarrow \begin{bmatrix} e^* & f^* & g^* & h^* \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \Rightarrow ae^* + bf^* + cg^* + dh^* = 0$$

Similarly,

$$\langle 00|01\rangle = 0 \Rightarrow \langle 00| V^\dagger V |01\rangle = 0 \Rightarrow \langle 01| \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Rightarrow b = 0 \tag{1}$$

Similarly,

$$\langle 00|11\rangle = 0 \Rightarrow \langle 00| V^\dagger V |11\rangle = 0 \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = 0 \Rightarrow f = 0 \tag{2}$$

Finally

$$\langle 10|01\rangle = 0 \Rightarrow \langle 10|\, V^\dagger V\, |01\rangle = 0 \Rightarrow \begin{bmatrix}\sqrt{\gamma} & 0 & 0 & \sqrt{1-\gamma}\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix} = 0$$

$$\Rightarrow a\sqrt{\gamma} + d\sqrt{1-\gamma} = 0 \Rightarrow d = -\frac{\sqrt{\gamma}}{\sqrt{1-\gamma}}a \tag{3}$$

In a similar way, we can write:

$$\begin{aligned}\langle 10|11\rangle &= 0\\ \Rightarrow \langle 10|\, V^\dagger V\, |11\rangle &= 0\\ \Rightarrow \begin{bmatrix}\sqrt{\gamma} & 0 & 0 & \sqrt{1-\gamma}\end{bmatrix}\begin{bmatrix}e\\ f\\ g\\ h\end{bmatrix} &= 0\\ \Rightarrow e\sqrt{\gamma} + h\sqrt{1-\gamma} = 0 \Rightarrow h = -\frac{\sqrt{\gamma}}{\sqrt{1-\gamma}}e\end{aligned}$$

From Eq. (1) and Eq. (2),

$$b = 0 \quad \text{and} \quad f = 0$$

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \Rightarrow |a|^2 + |c|^2 + \frac{\gamma}{1-\gamma}|a|^2 = 1 \Rightarrow |c|^2 = 1 - \frac{1}{1-\gamma}|a|^2$$

In a similar way:

$$|g|^2 = 1 - \frac{1}{1-\gamma}|e|^2$$

Now

$$ae^* + bf^* + cg^* + dh^* = 0$$

$$\Rightarrow ae^* + 0 + cg^* + \left(\frac{-\sqrt{\gamma}}{\sqrt{1-\gamma}}a\right)\cdot\left(\frac{-\sqrt{\gamma}}{\sqrt{1-\gamma}}e^*\right) = 0$$

Let us assume:

$$a = |a|e^{i\alpha} \Rightarrow d = \frac{-\sqrt{\gamma}}{\sqrt{1-\gamma}}|a|e^{i\alpha}$$

$$c = \sqrt{1 - \frac{1}{1-\gamma}|a|^2}\, e^{i\beta} \tag{5}$$

$$e = \sqrt{1 - \frac{1}{1-\gamma}|e|^2}\, e^{i\gamma} \tag{6}$$

If we opt $e = 0$, then $g = 1$ and opting:

$$a = -\sqrt{1-\gamma} \Rightarrow |a|^2 = 1-\gamma$$

$$c = \sqrt{1 - \frac{1}{1-\gamma}(1-\gamma)\, e^{i\beta}} = 0$$

$$d = \frac{-\sqrt{\gamma}}{\sqrt{1-\gamma}}\left(-\sqrt{1-\gamma}\right) = \sqrt{\gamma}$$

Here:

$$\begin{aligned} V: \quad & |00\rangle_{BE} \rightarrow |00\rangle_{AE} \\ & |01\rangle_{BE} \rightarrow \sqrt{1-\gamma}\,|00\rangle_{AE} + \sqrt{\gamma}\,|10\rangle_{AE} \\ & |10\rangle_{BE} \rightarrow \sqrt{\gamma}\,|00\rangle_{AE} + \sqrt{1-\gamma}\,|11\rangle_{AE} \\ & |11\rangle_{BE} \rightarrow |11\rangle_{AE} \end{aligned}$$

Thus, the solution is:

$$\begin{bmatrix} \langle 0|_B \langle 0|_E V |0\rangle_A \langle 0|_E V |0\rangle_A & \langle 0|_B \langle 0|_E V |0\rangle_A \langle 0|_E V |1\rangle_A & \langle 0|_B \langle 0|_E V |1\rangle_A \langle 0|_E V |0\rangle_A & \langle 0|_B \langle 0|_E V |1\rangle_A \langle 0|_E V |1\rangle_A \\ \langle 0|_B \langle 1|_E V |0\rangle_A \langle 0|_E V |0\rangle_A & \langle 0|_B \langle 1|_E V |0\rangle_A \langle 0|_E V |1\rangle_A & \langle 0|_B \langle 1|_E V |1\rangle_A \langle 0|_E V |0\rangle_A & \langle 0|_B \langle 1|_E V |1\rangle_A \langle 0|_E V |1\rangle_A \\ \langle 1|_B \langle 0|_E V |0\rangle_A \langle 0|_E V |0\rangle_A & \langle 1|_B \langle 0|_E V |0\rangle_A \langle 0|_E V |1\rangle_A & \langle 1|_B \langle 0|_E V |1\rangle_A \langle 0|_E V |0\rangle_A & \langle 1|_B \langle 0|_E V |1\rangle_A \langle 0|_E V |1\rangle_A \\ \langle 1|_B \langle 1|_E V |0\rangle_A \langle 0|_E V |0\rangle_A & \langle 1|_B \langle 1|_E V |0\rangle_A \langle 0|_E V |1\rangle_A & \langle 1|_B \langle 1|_E V |1\rangle_A \langle 0|_E V |0\rangle_A & \langle 1|_B \langle 1|_E V |1\rangle_A \langle 0|_E V |1\rangle_A \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & 0 & \sqrt{\gamma} & 0 \end{bmatrix}$$

Exercise 5.2.11 (Mark Wilde) Consider the full unitary operator for the amplitude damping channel from the previous exercise. Show that the density operator

$$\mathrm{Tr}_B \left\{ V(\rho_A \otimes |0\rangle\langle 0|_E) V^\dagger \right\} \tag{5.50}$$

that Eve receives has the following matrix representation:

$$\begin{bmatrix} \gamma p & \sqrt{\gamma}\eta^* \\ \sqrt{\gamma}\eta & 1-\gamma p \end{bmatrix} \quad \text{if} \quad \rho_A = \begin{bmatrix} 1-p & \eta \\ \eta^* & p \end{bmatrix}. \tag{5.51}$$

By comparing with (4.356), observe that the output to Eve is the bit flip of the output of an amplitude damping channel with damping parameter $1-\gamma$.

Solution:

To compute $\mathrm{Tr}_B \left\{ V \left(\rho_A \otimes |0\rangle\langle 0|_E \right) V^\dagger \right\}$ we have to evaluate the partial trace

$$\mathrm{Tr}_B \left\{ V \left(\rho_A \otimes |0\rangle\langle 0|_E \right) V^\dagger \right\} = \sum_x \langle x|_B \otimes I_E \, V \left(\rho_A \otimes |0\rangle\langle 0|_E \right) V^\dagger \; |x\rangle_B \otimes I_E$$

From the previous question we know that the matrix representation of V is

$$V = \sum_{i,j,k,l} \alpha_{ijkl} \, |i\rangle_B \, |j\rangle_E \, \langle k|_A \, \langle l|_E$$

then

$$V^\dagger = \sum_{m,n,p,q} \left(\alpha_{mnpq} \left|m\right\rangle_B \left|n\right\rangle_E \left\langle p\right|_A \left\langle q\right|_E\right)^\dagger = \sum_{m,n,p,q} \alpha^*_{mnpq} \left|p\right\rangle_A \left|q\right\rangle_E \left\langle m\right|_B \left\langle n\right|_E$$

We already have computed

$$V\left(\rho_A \otimes |0\rangle\!\langle 0|_E\right) V^\dagger = \sum_{i,j,k} \sum_{m,n,p} \alpha_{ijk0}\alpha^*_{mnp0} \left|i\right\rangle_B \left|j\right\rangle_E \left\langle k\right|_A \rho_A \left|p\right\rangle_A \left\langle m\right|_B \left\langle n\right|_E \tag{1}$$

Therefore, the partial trace we can evaluate as

$$\begin{aligned}
&\mathrm{Tr}_B \left\{ V(\rho_A \otimes |0\rangle\!\langle 0|_E) V^\dagger \right\} \\
&= \sum_x \left\langle x\right|_B \otimes I_E \left\{ \sum_{i,j,k} \sum_{m,n,p} \alpha_{ijk0}\alpha^*_{mnp0} \left|i\right\rangle_B \left|j\right\rangle_E \left\langle k\right|_A \rho_A \left|p\right\rangle_A \left\langle m\right|_B \left\langle n\right|_E \right\} \left|x\right\rangle_B \otimes I_E \\
&= \sum_x \sum_{i,j,k} \sum_{m,n,p} \alpha_{ijk0}\alpha^*_{mnp0} \underbrace{\langle x\rangle\, i_B}_{\delta_{x,i}} \left|j\right\rangle_E \left\langle k\right|_A \rho_A \left|p\right\rangle_A \underbrace{\langle m\rangle\, x_B}_{\delta_{x,m}} \left\langle n\right|_E \\
&= \sum_{j,k} \sum_{n,p} \alpha_{xjk0}\alpha^*_{xnp0} \left|j\right\rangle_E \left\langle k\right|_A \rho_A \left|p\right\rangle_A \left\langle n\right|_E
\end{aligned} \tag{2}$$

From the previous question **Exercise 5.2.10** we have the coefficients

$$\begin{aligned}
&\alpha_{0001} = -\sqrt{1-\gamma}, \quad \alpha_{0010} = \sqrt{\gamma}, \quad \alpha_{0100} = 1, \quad \alpha_{1011} = 1, \\
&\alpha_{0101} = 0, \quad \alpha_{1101} = \sqrt{\gamma}, \quad \alpha_{1110} = \sqrt{1-\gamma}
\end{aligned}$$

Rest of the elements of V are zero. Also, since no elements with $l = 1$ appear in Equation (2), we need to consider only

$$\alpha_{0010} = \sqrt{\gamma}, \quad \alpha_{0100} = 1, \quad \text{and} \quad \alpha_{1110} = \sqrt{1-\gamma}$$

Applying all the coefficents in equation (3) becomes

$$\begin{aligned}
&\alpha_{0010}\alpha^*_{0010} \left|0\right\rangle_E \left\langle 1\right|_A \rho_A \left|1\right\rangle_A \left\langle 0\right|_E + \alpha_{0010}\alpha^*_{0100} \left|0\right\rangle_E \left\langle 1\right|_A \rho_A \left|0\right\rangle_A \left\langle 1\right|_E \\
&+ \alpha_{0100}\alpha^*_{0010} \left|1\right\rangle_E \left\langle 0\right|_A \rho_A \left|1\right\rangle_A \left\langle 0\right|_E + \alpha_{0100}\alpha^*_{0100} \left|1\right\rangle_E \left\langle 0\right|_A \rho_A \left|0\right\rangle_A \left\langle 1\right|_E \\
&+ \alpha_{1110}\alpha^*_{1110} \left|1\right\rangle_E \left\langle 1\right|_A \rho_A \left|1\right\rangle_A \left\langle 1\right|_E \\
&= \alpha^2_{0010}(\rho_A)_{11} \left|0\right\rangle_E \left\langle 0\right| + \alpha_{0010}\alpha_{0100}(\rho_A)_{10} \left|0\right\rangle_E \left\langle 1\right| \\
&\quad + \alpha_{0100}\alpha_{0010}(\rho_A)_{01} \left|1\right\rangle_E \left\langle 0\right| + \alpha^2_{0100}(\rho_A)_{00} \left|1\right\rangle_E \left\langle 1\right| + \alpha^2_{1110}(\rho_A)_{11} \left|1\right\rangle_E \left\langle 1\right|
\end{aligned} \tag{3}$$

Given, $\rho_A = \begin{bmatrix} 1-p & \eta \\ \eta^* & p \end{bmatrix}$ Putting ρ_A in the Equation (3) we get

$$\begin{aligned}
&\gamma p \left|10\right\rangle_E \left\langle 01\right| + \sqrt{\gamma}\eta^* \left|10\right\rangle_E \left\langle 11\right| + \sqrt{\gamma}\eta \left|11\right\rangle_E \left\langle 01\right| + (1-p) \left|11\right\rangle_E \left\langle 11\right| \\
&= \gamma p \left|0\right\rangle_E \left\langle 0\right| + \sqrt{\gamma}\eta^* \left|0\right\rangle_E \left\langle 1\right| + \sqrt{\gamma}\eta \left|1\right\rangle_E \left\langle 0\right| + (1-\gamma p) \left|1\right\rangle_E \left\langle 1\right| \\
&= \gamma p \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \sqrt{\gamma}\eta^* \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \sqrt{\gamma}\eta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (1-\gamma p) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \gamma p & \sqrt{\gamma}\eta^* \\ \sqrt{\gamma}\eta & 1-\gamma p \end{bmatrix}
\end{aligned}$$

which is the required result.

Exercise 5.4.1 (Mark Wilde) Suppose that there is a set of density operators ρ_S^k and a POVM $\{\Lambda_S^k\}$ that identifies these states with high probability, in the sense that

$$\forall k \quad \operatorname{Tr}\left\{\Lambda_S^k \rho_S^k\right\} \geq 1-\varepsilon,$$

where $\varepsilon \in (0,1)$. *Construct a coherent measurement* $U_{S\to SS'}$ *and show that the coherent measurement has a high probability of success in the sense that*

$$\left|\langle\phi_k|_{RS}\,\langle k|_{S'}\, U_{S\to SS'}\,|\phi_k\rangle_{RS}\right|^2 \geq 1-\varepsilon,$$

where each $|\phi_k\rangle_{RS}$ is a purification of ρ_k.

Solution: Given: $\forall k$,

$$\operatorname{tr}\left(\Lambda_S^k \rho_S^k\right) \geq 1-\epsilon, \quad \epsilon \in (0,1)$$

We have to find a coherent measurement $U_{S\to SS'}$ such that:

$$\left|\langle\phi_k|_{RS}\,\langle k|_S'\, U_{S\to SS'}\,|\phi_k\rangle_{RS}\rangle\right|^2 \geq 1-\varepsilon$$

Let

$$U_{S\to SS'} = \sum_j \left(I_R \otimes \Lambda_S^j\right) \otimes |j\rangle_{S'}$$

Therefore,

$$\left|\langle\phi_k|_{RS}\,\langle k|_S'\, U_{S\to SS'}\,|\phi_k\rangle_{RS}\rangle\right|^2$$

$$= \left|\langle\phi_k|_{RS}\,\langle k|_S' \left(\sum_j \left(I_R \otimes \Lambda_S^j\right) \otimes |j\rangle_{S'}\,\langle j|_S\right) |\phi_k\rangle_{RS}\right|^2$$

Since the quantity above is a scalar, we get:

$$= \left|\operatorname{Tr}_{RS}\left\{\langle\phi_k|_{RS}\,\langle k|_S' \left(\sum_j \left(I_R \otimes \Lambda_S^j\right) \otimes |j\rangle_{S'}\,\langle j|_S\right) |\phi_k\rangle_{RS}\right\}\right|^2$$

$$= \left|\operatorname{Tr}_{RS}\left\{\sum_j \langle\phi_k|_{RS}\left(I_R \otimes \Lambda_S^j\right)|\phi_k\rangle_{RS}\,\langle k|j\rangle_S'\right\}\right|^2$$

$$= \left|\operatorname{Tr}_{RS}\left\{\langle\phi_k|_{RS}\left(I_R \otimes \Lambda_S^k\right)|\phi_k\rangle_{RS}\right\}\right|^2$$

By cyclic property of trace, we obtain:

$$= \left|\operatorname{Tr}_{RS}\left\{\left(I_R \otimes \Lambda_S^k\right)|\phi_k\rangle_{RS}\,\langle\phi_k|\right\}\right|^2$$

By using the linearity of trace:

$$= \left|\operatorname{Tr}_S\left\{\Lambda_S \operatorname{Tr}_R\left(|\phi_k\rangle_{RS}\,\langle\phi_k|\right)\right\}\right|^2$$

$$= \left|\mathrm{Tr}_S\left\{\Lambda_S^k\left(S_S^k\right)_{\text{purified}}\right\}\right|^2 = \left|\mathrm{Tr}_S\left\{\Lambda_S^k S_S^k\right\}\right|^2 \geq 1-\epsilon$$

where $(\rho_S)_{\text{purified}}$ is the purification of ρ_S.

$$\boxed{\left|\langle\phi_k|_{RS}\,\langle k|_{S'}\,U_{S\to SS'}\,|\phi_k\rangle_{RS}\right|^2 \geq 1-\varepsilon}$$

Exercise 9.2.7 (Mark Wilde) Let $\rho, \sigma \in \mathcal{D}(\mathcal{H})$. Show that the fidelity is invariant with respect to an isometry $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}_0)$, i.e.,

$$F(\rho,\sigma) = F(U\rho U^\dagger, U\sigma U^\dagger).$$

Solution:

To show that fidelity is invariant under an isometry U, we start with the definition $F(\rho,\sigma) = \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1^2$. If U is an isometry (i.e., $U^\dagger U = I$), then $\sqrt{U\rho U^\dagger} = U\sqrt{\rho}U^\dagger$, and similarly for σ. So we compute:

$$F(U\rho U^\dagger, U\sigma U^\dagger) = \left\|\sqrt{U\rho U^\dagger}\sqrt{U\sigma U^\dagger}\right\|_1^2 = \left\|U\sqrt{\rho}\sqrt{\sigma}U^\dagger\right\|_1^2.$$

Using the fact that the trace norm is invariant under isometries, we get $\left\|U\sqrt{\rho}\sqrt{\sigma}U^\dagger\right\|_1 = \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1$. Therefore, $F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho,\sigma)$, as required.

Exercise 9.2.8 (Mark Wilde) Let $\rho, \sigma \in \mathcal{D}(\mathcal{H}_A)$ and let $\mathcal{N} : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$ be a quantum channel. Show that the fidelity is monotone with respect to the channel $\mathcal{N}$, i.e.,

$$F(\rho,\sigma) \leq F(\mathcal{N}(\rho), \mathcal{N}(\sigma)).$$

Solution: The fidelity between two density matrices ρ and σ is given by:

$$F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2.$$

Uhlmann's theorem states that:

$$F(\rho,\sigma) = \max_{|\psi_\rho\rangle,|\psi_\sigma\rangle} |\langle\psi_\rho|\psi_\sigma\rangle|^2,$$

where $|\psi_\rho\rangle, |\psi_\sigma\rangle$ are purifications of ρ and σ, respectively.

Let $|\psi_\rho\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R$ be a purification of ρ, and similarly $|\psi_\sigma\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R$ a purification of σ. Then, the action of a quantum channel $\mathcal{N}$ on subsystem A gives rise to a new state:

$$(\mathcal{N} \otimes \mathbb{I}_R)(|\psi_\rho\rangle\langle\psi_\rho|),$$

which is a purification of $\mathcal{N}(\rho)$, and similarly for σ.

Since $\mathcal{N} \otimes \mathbb{I}$ is a completely positive trace-preserving (CPTP) map, it preserves inner products in the sense that it cannot increase the fidelity. Hence:

$$|\langle\psi_\rho|\psi_\sigma\rangle|^2 \leq |\langle\psi'_\rho|\psi'_\sigma\rangle|^2,$$

where $\psi' = (\mathcal{N} \otimes \mathbb{I})(\psi)$.

From Uhlmann's theorem and the above observation:

$$F(\rho, \sigma) = \max_{\psi_\rho, \psi_\sigma} |\langle \psi_\rho | \psi_\sigma \rangle|^2 \leq \max_{\psi_\rho, \psi_\sigma} |\langle (\mathcal{N} \otimes \mathbb{I})\psi_\rho | (\mathcal{N} \otimes \mathbb{I})\psi_\sigma \rangle|^2 = F(\mathcal{N}(\rho), \mathcal{N}(\sigma)).$$

Conclusion:

$$F(\rho, \sigma) \leq F(\mathcal{N}(\rho), \mathcal{N}(\sigma)),$$

so the fidelity is monotonic under the action of a quantum channel $\mathcal{N}$.

2. **PROBLEM 2:**
Show that the trace distance between two density operators is equivalent to the Euclidean distance between their respective Bloch vectors. Interpret your results.

Solution:

For two single-qubit density operators ρ and σ with corresponding Bloch vectors $\vec{r}$ and $\vec{s}$, we can show that the trace distance between them equals half the Euclidean distance between their Bloch vectors.

First, we express the density operators in terms of their Bloch vectors:

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}), \quad \sigma = \frac{1}{2}(I + \vec{s} \cdot \vec{\sigma}),$$

where I is the identity matrix and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.

The trace distance is defined as:

$$T(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1.$$

Computing the difference:

$$\rho - \sigma = \frac{1}{2}(\vec{r} - \vec{s}) \cdot \vec{\sigma}.$$

The eigenvalues of $(\vec{r} - \vec{s}) \cdot \vec{\sigma}$ are $\pm|\vec{r} - \vec{s}|$, so the trace norm becomes:

$$\|\rho - \sigma\|_1 = |\vec{r} - \vec{s}|.$$

Thus, the trace distance simplifies to:

$$T(\rho, \sigma) = \frac{1}{2}|\vec{r} - \vec{s}|.$$

This shows that the trace distance between two single-qubit states is exactly half the Euclidean distance between their Bloch vectors. This result has a clear geometric interpretation in the Bloch sphere representation: states that are farther apart in the Bloch sphere (larger Euclidean distance between their vectors) are more distinguishable (larger trace distance). When two states are antipodal (like $|0\rangle$ and $|1\rangle$), their

Bloch vectors are maximally separated ($|\vec{r}-\vec{s}|=2$) and the trace distance reaches its maximum value of 1.

The key result is:

$$\boxed{T(\rho,\sigma)=\frac{1}{2}|\vec{r}-\vec{s}|}$$

3. **PROBLEM 3: Exercise 9.13 (Neilson Chuang)** Show that the bit flip channel is contractive but not strictly contractive. Find the set of fixed points.

Solution:

The bit flip channel is defined as:

$$\mathcal{E}(\rho)=(1-p)\rho+pX\rho X,$$

where $0<p<1$, and X is the Pauli-X operator.

Contractive It is a well-known fact that any completely positive trace-preserving (CPTP) map is contractive with respect to the trace distance. That is,

$$D(\mathcal{E}(\rho),\mathcal{E}(\sigma))\leq D(\rho,\sigma)$$

for all density matrices ρ and σ. Since the bit flip channel is a CPTP map, it is contracting. Let $\mathcal{E}$ be a completely positive trace-preserving (CPTP) map, and let ρ and σ be two density operators. The trace distance between ρ and σ is defined as:

$$D(\rho,\sigma)=\frac{1}{2}\|\rho-\sigma\|_1,$$

where the trace norm is defined by:

$$\|A\|_1=\mathrm{Tr}\left(\sqrt{A^\dagger A}\right).$$

We want to prove that:

$$D(\mathcal{E}(\rho),\mathcal{E}(\sigma))\leq D(\rho,\sigma).$$

This result follows from the monotonicity of the trace norm under CPTP maps. According to the Stinespring dilation theorem, any CPTP map $\mathcal{E}$ can be represented as:

$$\mathcal{E}(\rho)=\mathrm{Tr}_E\left(U(\rho\otimes|0\rangle\langle 0|)U^\dagger\right),$$

for some unitary U on a larger Hilbert space and partial trace over an environment E. Since the trace norm is unitarily invariant:

$$\|UAU^\dagger\|_1=\|A\|_1,$$

and non-increasing under partial trace:

$$\|\mathrm{Tr}_E(A)\|_1\leq\|A\|_1,$$

we have:

$$\|\mathcal{E}(\rho) - \mathcal{E}(\sigma)\|_1 \leq \|\rho - \sigma\|_1.$$

Therefore,

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma),$$

which proves that CPTP maps are contractive with respect to the trace distance.

Not Strictly Contractive A channel is strictly contractive if:

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma) \quad \text{for all } \rho \neq \sigma.$$

Let $\rho = \frac{1}{2}(I + aX)$, $\sigma = \frac{1}{2}(I + bX)$. Since $X\rho X = \rho$ and similarly for σ, we find:

$$\mathcal{E}(\rho) = \rho, \quad \mathcal{E}(\sigma) = \sigma,$$

so

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = D(\rho, \sigma),$$

which shows that the channel is not strictly contractive. **Fixed Points** We look for density matrices ρ such that:

$$\mathcal{E}(\rho) = \rho.$$

This implies:

$$(1 - p)\rho + pX\rho X = \rho \quad \Rightarrow \quad X\rho X = \rho.$$

Thus, ρ must be invariant under conjugation by X. This happens if and only if ρ is of the form:

$$\rho = \frac{1}{2}(I + aX), \quad \text{with } -1 \leq a \leq 1.$$

Therefore, the set of fixed points is the set of all density matrices lying along the x-axis of the Bloch sphere.

Exercise 9.9 (Neilson Chuang) Existence of Fixed Points for Quantum Operations

Solution:

We have to prove that any trace-preserving quantum operation $\mathcal{E}$ has a fixed point, i.e., there exists a density operator ρ such that

$$\mathcal{E}(\rho) = \rho.$$

Approach: Use Schauder's fixed point theorem.

Schauder's fixed point theorem: *Let K be a convex, compact subset of a Hilbert space, and let $f : K \to K$ be a continuous map. Then f has at least one fixed point.*

Application to Quantum Channels:

Let $\mathcal{D}(\mathcal{H})$ denote the set of all density operators on a finite-dimensional Hilbert space $\mathcal{H}$. This set has the following properties:

- **Convex:** If $\rho_1, \rho_2 \in \mathcal{D}(\mathcal{H})$, then for any $\lambda \in [0,1]$, the operator $\lambda\rho_1 + (1-\lambda)\rho_2$ is also a density operator.
- **Compact:** In finite-dimensional spaces, the set of density matrices is closed and bounded in the space of trace-class operators.
- **Subset of a Hilbert space:** Density matrices are elements of the space of Hermitian operators, which forms a finite-dimensional Hilbert space.
- **Continuity:** Any trace-preserving quantum operation $\mathcal{E}$ is a completely positive trace-preserving (CPTP) linear map and thus continuous.

Therefore, all conditions of Schauder's fixed point theorem are satisfied.

Conclusion: There exists at least one density operator ρ such that

$$\mathcal{E}(\rho) = \rho.$$

Exercise 9.11 (Nielsen and Chuang) Suppose $\mathcal{E}$ is a trace-preserving quantum operation for which there exists a density operator ρ_0 and a trace-preserving quantum operation $\mathcal{E}'$ such that

$$\mathcal{E}(\rho) = p\rho_0 + (1-p)\mathcal{E}'(\rho), \tag{9.52}$$

for some p, $0 < p \leq 1$. Physically, this means that with probability p the input state is thrown out and replaced with the fixed state ρ_0, while with probability $1-p$ the operation $\mathcal{E}'$ occurs. Use joint convexity to show that $\mathcal{E}$

Solution:

Given the trace-preserving quantum operation $\mathcal{E}$ with the decomposition:

$$\mathcal{E}(\rho) = p\rho_0 + (1-p)\mathcal{E}'(\rho),$$

where $0 < p \leq 1$, ρ_0 is a fixed density operator, and $\mathcal{E}'$ is another trace-preserving quantum operation, we will prove that $\mathcal{E}$ is strictly contractive and consequently has a unique fixed point.

For any two density operators ρ and σ, we analyze the trace distance after applying $\mathcal{E}$:

$$T(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = T\left(p\rho_0 + (1-p)\mathcal{E}'(\rho), p\rho_0 + (1-p)\mathcal{E}'(\sigma)\right).$$

By the joint convexity of the trace distance, this satisfies:

$$T(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq p\,T(\rho_0, \rho_0) + (1-p)\,T(\mathcal{E}'(\rho), \mathcal{E}'(\sigma)) = (1-p)\,T(\mathcal{E}'(\rho), \mathcal{E}'(\sigma)),$$

since $T(\rho_0, \rho_0) = 0$. Furthermore, because any quantum operation is contractive under the trace distance, we have:

$$T(\mathcal{E}'(\rho), \mathcal{E}'(\sigma)) \leq T(\rho, \sigma).$$

Combining these inequalities yields the strict contractivity:

$$T(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq (1-p)\,T(\rho, \sigma) < T(\rho, \sigma),$$

where the strict inequality holds because $0 < p \leq 1$ ensures $0 \leq 1 - p < 1$.

To establish the existence and uniqueness of the fixed point, consider the sequence $\rho_{n+1} = \mathcal{E}(\rho_n)$ starting from any initial state ρ_1. The contractivity implies:

$$T(\rho_{n+1}, \rho_n) \leq (1-p)\, T(\rho_n, \rho_{n-1}) \leq (1-p)^n T(\rho_1, \rho_0).$$

Since $(1-p)^n \to 0$ as $n \to \infty$, the sequence is Cauchy. The space of density operators being complete under the trace distance guarantees convergence to a unique fixed point ρ_* satisfying $\mathcal{E}(\rho_*) = \rho_*$.

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