Quantum Information Theory (E2-270) (Spring 2025) **HW2**
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1. **PROBLEM 1: Exercise 4.6.3 (Mark Wilde)** Show that both a classical–quantum channel and a quantum–classical channel are entanglement-breaking—i.e., if we input the A system of a bipartite state ρ_{RA} to either of these channels, then the resulting state on systems RB is separable.
Solution:

A quantum channel $\mathcal{N}_{A\to B}$ is entanglement-breaking if for every bipartite input state ρ_{RA}, the resulting state

$$(\mathcal{I}_R \otimes \mathcal{N}_{A\to B})(\rho_{RA})$$

is separable across the $R : B$ partition. This means it can be written as:

$$\rho_{RB} = \sum_i p_i \rho_R^{(i)} \otimes \rho_B^{(i)}$$

for some probability distribution $\{p_i\}$ and states $\rho_R^{(i)}, \rho_B^{(i)}$. A sufficient condition for $\mathcal{N}_{A\to B}$ to be entanglement-breaking if it can be expressed as:

$$\mathcal{N}_{A\to B}(\rho) = \sum_i B_i \rho B_i^\dagger,$$

where the Kraus operators B_i are rank-1, meaning they map any input state to a pure or classical state.

A classical–quantum channel (CQ) is a quantum channel that first measures a quantum state and then encodes the classical outcome into a fixed set of quantum states. It can be written as:

$$\mathcal{N}_{CQ}(\rho) = \sum_i \mathrm{Tr}(M_i \rho) \left|\psi_i\right\rangle \left\langle\psi_i\right|,$$

where $\{M_i\}$ is a positive operator-valued measure (POVM) representing a measurement, and $|\psi_i\rangle$ are fixed quantum states assigned to each outcome. When applied to one half of an entangled state ρ_{RA}, the resulting bipartite state is:

$$\rho_{RB} = \sum_i \mathrm{Tr}_A(M_i \rho_{RA}) \otimes \left|\psi_i\right\rangle \left\langle\psi_i\right|.$$

Since this is an explicit convex sum of product states, it is separable, proving that a CQ channel is entanglement-breaking.

A quantum–classical (QC) channel measures a quantum system and outputs classical information. It is described by:

$$\mathcal{N}_{QC}(\rho) = \sum_i \mathrm{Tr}(M_i \rho) \left|i\right\rangle \left\langle i\right|,$$

where $|i\rangle$ are orthonormal classical basis states. For a bipartite input ρ_{RA}, the output state is:

$$\rho_{RB} = \sum_i \mathrm{Tr}_A(M_i \rho_{RA}) \otimes |i\rangle \langle i| \,.$$

This is again a classical mixture of separable states, proving that a QC channel is also entanglement-breaking.

Since both CQ and QC channels produce output states that are convex combinations of separable states, they always destroy entanglement when acting on part of an entangled state. Therefore, both CQ and QC channels are entanglement-breaking.

Exercise 4.7.5 (Mark Wilde) Show that the action of a depolarizing channel on the Bloch vector is

$$(r_x, r_y, r_z) \to ((1-p)r_x, (1-p)r_y, (1-p)r_z).$$

Thus, it uniformly shrinks the Bloch vector to become closer to the maximally mixed state.

Solution:

The depolarizing channel $\mathcal{N}_{\text{dep}}$ is a quantum noise channel that replaces the input quantum state ρ with the maximally mixed state $I/2$ with probability p, while leaving it unchanged with probability $1-p$. Mathematically, this channel acts on a quantum state ρ as:

$$\mathcal{N}_{\text{dep}}(\rho) = (1-p)\rho + \frac{p}{2} I.$$

Any single-qubit density matrix ρ can be expressed in terms of its Bloch vector components r_x, r_y, r_z as:

$$\rho = \frac{1}{2}\left(I + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z\right),$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. The depolarizing channel acts linearly on ρ, so applying it to the above representation gives:

$$\mathcal{N}_{\text{dep}}(\rho) = (1-p)\rho + \frac{p}{2} I.$$

Substituting the expression for ρ:

$$\mathcal{N}_{\text{dep}}(\rho) = (1-p) \cdot \frac{1}{2}\left(I + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z\right) + \frac{p}{2} I.$$

Expanding and simplifying,

$$\mathcal{N}_{\text{dep}}(\rho) = \frac{1}{2}\left(I + (1-p) r_x \sigma_x + (1-p) r_y \sigma_y + (1-p) r_z \sigma_z\right).$$

From this, we see that the Bloch vector transforms as:

$$(r_x, r_y, r_z) \to ((1-p)r_x, (1-p)r_y, (1-p)r_z).$$

This shows that the depolarizing channel **uniformly shrinks the Bloch vector** by a factor of $(1-p)$, effectively bringing the quantum state closer to the maximally mixed state $I/2$. When $p=1$, all the Bloch vector components vanish, leaving the completely mixed state $I/2$, which has no coherence or purity. The depolarizing channel thus serves as a model of isotropic noise affecting a quantum system, gradually erasing information about the original state.

Exercise: 4.7.6 (Mark Wilde) Show that randomly applying the Heisenberg–Weyl operators

$$\{X(i)Z(j)\}_{i,j\in\{0,\dots,d-1\}}$$

with uniform probability to any qudit density operator gives the maximally mixed state π:

$$\frac{1}{d^2}\sum_{i,j=0}^{d-1} X(i)Z(j)\rho Z^\dagger(j)X^\dagger(i) = \pi.$$

(Hint: You can do the full calculation, or you can decompose this channel into the composition of two completely dephasing channels where the first is a dephasing in the computational basis and the next is a dephasing in the conjugate basis.)

Solution:

The Heisenberg–Weyl operators $X(i)$ and $Z(j)$ are the generalized Pauli operators for a d-dimensional qudit system, defined as:

$$X(j)\left|k\right\rangle = \left|k+j \mod d\right\rangle, \quad Z(j)\left|k\right\rangle = e^{2\pi ijk/d}\left|k\right\rangle.$$

These operators satisfy the commutation relation:

$$Z(j)X(i) = e^{2\pi ij/d}X(i)Z(j).$$

The given quantum channel applies these operators uniformly over all i,j, meaning it performs an average over all possible displacements in the qudit Hilbert space. Mathematically, this channel is represented as:

$$\mathcal{N}(\rho) = \frac{1}{d^2}\sum_{i,j=0}^{d-1} X(i)Z(j)\rho Z^\dagger(j)X^\dagger(i).$$

Effect of Averaging over Heisenberg–Weyl Operators To understand the effect of $\mathcal{N}$ on a general density matrix ρ, we expand ρ in terms of the Heisenberg–Weyl basis:

$$\rho = \sum_{m,n} c_{m,n}X(m)Z(n).$$

Since the channel sums over all $X(i)Z(j)$, let's analyze how these terms transform:

$$X(i)Z(j)X(m)Z(n)Z^\dagger(j)X^\dagger(i) = \omega^{m(j-n)-in}Z(n)X(m).$$

Using the commutation relation, this results in a phase factor depending on i, j, m, n, which averages to zero unless $m = n = 0$. The only surviving term is the identity matrix contribution:

$$\frac{1}{d^2}\sum_{i,j} X(i)Z(j)\rho Z^\dagger(j)X^\dagger(i) = \frac{I}{d}.$$

Alice then transmits two classical bits encoding the measurement result to Bob. Upon receiving this information, Bob applies the appropriate Pauli correction I, X, Z, XZ to recover $|\psi\rangle$ perfectly.

Exercise 4.7.8 (Mark Wilde): Show that the amplitude damping channel obeys a composition rule. Consider an amplitude damping channel $\mathcal{N}_1$ with transmission parameter $(1-\gamma_1)$ and another amplitude damping channel $\mathcal{N}_2$ with transmission parameter $(1-\gamma_2)$. Show that the composition channel $\mathcal{N}_2 \circ \mathcal{N}_1$ is an amplitude damping channel with transmission parameter $(1-\gamma_2)(1-\gamma_1)$. (Note that the transmission parameter is equal to one minus the damping parameter.)

Solution: The amplitude damping channel models energy dissipation in a quantum system, such as spontaneous emission. The channel is described by the Kraus operators:

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where γ represents the damping probability.

For the first amplitude damping channel $\mathcal{N}_1$ with damping parameter γ_1, the Kraus operators are:

$$E_0^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma_1} \end{bmatrix}, \quad E_1^{(1)} = \begin{bmatrix} 0 & \sqrt{\gamma_1} \\ 0 & 0 \end{bmatrix}. \tag{2}$$

Applying another amplitude damping channel $\mathcal{N}_2$ with damping parameter γ_2 leads to new Kraus operators $E_0^{(2)}$ and $E_1^{(2)}$:

$$E_0^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma_2} \end{bmatrix}, \quad E_1^{(2)} = \begin{bmatrix} 0 & \sqrt{\gamma_2} \\ 0 & 0 \end{bmatrix}. \tag{3}$$

The composition of the two channels is given by applying the Kraus operators of $\mathcal{N}_2$ after those of $\mathcal{N}_1$. The effective Kraus operators are:

$$E_0^{(2)}E_0^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{(1-\gamma_1)(1-\gamma_2)} \end{bmatrix}, \tag{4}$$

$$E_0^{(2)}E_1^{(1)} = \begin{bmatrix} 0 & \sqrt{\gamma_1(1-\gamma_2)} \\ 0 & 0 \end{bmatrix}, \tag{5}$$

$$E_1^{(2)}E_0^{(1)} = \begin{bmatrix} 0 & \sqrt{\gamma_2(1-\gamma_1)} \\ 0 & 0 \end{bmatrix}, \tag{6}$$

$$E_1^{(2)}E_1^{(1)} = \begin{bmatrix} 0 & \sqrt{\gamma_1\gamma_2} \\ 0 & 0 \end{bmatrix}. \tag{7}$$

The new Kraus operators describe an amplitude damping channel with an effective damping parameter γ', given by the total probability of an excitation being lost:

$$\gamma' = 1 - (1 - \gamma_1)(1 - \gamma_2). \tag{8}$$

Thus, the transmission parameter of the composed channel is:

$$(1 - \gamma') = (1 - \gamma_1)(1 - \gamma_2). \tag{9}$$

This confirms that the composition of two amplitude damping channels is itself an amplitude damping channel with the expected transmission parameter, completing the proof.

2. **PROBLEM 2:** Work out the following problems:
(1) Establish the Schmidt decomposition result when the dimension of the quantum systems A and B are not the same, i.e., in the most general form.
Solution: The Schmidt decomposition theorem states that any pure state in a bipartite Hilbert space can be expressed as a sum of product states with nonnegative singular values. We establish this result in the general case where the subsystems A and B have different dimensions.

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces of dimensions d_A and d_B, respectively. Consider a pure state $|\psi\rangle$ in the composite Hilbert space:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B,$$

where $\dim(\mathcal{H}_A) = d_A$ and $\dim(\mathcal{H}_B) = d_B$, with possibly $d_A \neq d_B$.

Let $\{|e_i\rangle\}$ and $\{|f_j\rangle\}$ be orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. We can expand $|\psi\rangle$ as

$$|\psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} C_{ij} \, |e_i\rangle \otimes |f_j\rangle$$

where C is a $d_A \times d_B$ complex coefficient matrix.

The Schmidt decomposition follows from the singular value decomposition (SVD) of the coefficient matrix C. Using SVD, we can write C as

$$C = U \Lambda V^\dagger$$

where U is a $d_A \times d_A$ unitary matrix, V is a $d_B \times d_B$ unitary matrix, Λ is a $d_A \times d_B$ diagonal matrix with nonnegative singular values λ_k on the diagonal (arranged in non-increasing order). The number of nonzero singular values is at most $r = \min(d_A, d_B)$, which is the rank of C. Defining new orthonormal bases:

$$|\tilde{e}_k\rangle = \sum_{i=1}^{d_A} U_{ik} \, |e_i\rangle \,, \quad |\tilde{f}_k\rangle = \sum_{j=1}^{d_B} V_{jk} \, |f_j\rangle$$

we rewrite $|\psi\rangle$ as:

$$|\psi\rangle = \sum_{k=1}^{r} \lambda_k |\tilde{e}_k\rangle \otimes |\tilde{f}_k\rangle$$

where λ_k are the nonzero singular values of C, and $|\tilde{e}_k\rangle$, $|\tilde{f}_k\rangle$ form new orthonormal bases for subspaces of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. This is the general form of the Schmidt decomposition for a bipartite quantum state when the dimensions of the two subsystems are different. The number of nonzero terms in the sum is given by the Schmidt rank $r = \min(d_A, d_B)$, and the coefficients λ_k (Schmidt coefficients) determine the degree of entanglement of the state.

(2) Establish mathematically how Schmidt decomposition can help examine if a pure bipartite state $|\phi\rangle^{AB}$ is an entangled state or a product state.

Solution: Given a pure bipartite state $|\phi\rangle^{AB}$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the Schmidt decomposition theorem states that it can be written as

$$|\phi\rangle^{AB} = \sum_{k=1}^{r} \lambda_k |\tilde{e}_k\rangle \otimes |\tilde{f}_k\rangle$$

where λ_k are the Schmidt coefficients, $|\tilde{e}_k\rangle$ and $|\tilde{f}_k\rangle$ are orthonormal bases for subspaces of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and $r = \min(d_A, d_B)$ is the Schmidt rank of the state. To determine whether $|\phi\rangle^{AB}$ is an entangled state or a product state, we examine the Schmidt rank r: - If $r = 1$, then the state can be written as $|\phi\rangle^{AB} = \lambda_1 |\tilde{e}_1\rangle \otimes |\tilde{f}_1\rangle$, which is a product state, meaning there is no entanglement.
- If $r > 1$, then the state is entangled since it cannot be factorized into a tensor product of states from $\mathcal{H}_A$ and $\mathcal{H}_B$.

Comment: If the composite state $\rho^{AB} = |\psi\rangle\langle\psi|$ is pure, it is a product state if and only if the reduced density matrices ρ^A and ρ^B are pure states. For example, if Alice's spin is definitely up and Bob's spin is definitely down, then the composite state represents a pure state:

$$\rho_{\uparrow\downarrow} = |\uparrow\rangle_A \langle\uparrow| \otimes |\downarrow\rangle_B \langle\downarrow|,$$

where $|\uparrow\rangle = |0\rangle$ and $|\downarrow\rangle = |1\rangle$.

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Conversely, this means that every pure state whose subsystems are in mixed states must be entangled! [1]

3. **PROBLEM 3 : Exercise 8.3 (Nielsen and Chuang):** Our derivation of the operator-sum representation implicitly assumed that the input and output spaces for the operation were the same. Suppose a composite system AB initially in an unknown quantum state ρ is brought into contact with a composite system CD initially in some

standard state $|0\rangle$, and the two systems interact according to a unitary interaction U. After the interaction, we discard systems A and D, leaving a state $E(\rho)$ of system BC. Show that the map

$$E(\rho) = \sum_k E_k \rho E_k^\dagger \tag{10}$$

for some set of linear operators E_k from the state space of system AB to the state space of system BC, and such that

$$\sum_k E_k^\dagger E_k = I. \tag{11}$$

Solution:

We begin with a composite system AB in an unknown quantum state ρ and another system CD initialized in a standard state $|0\rangle$. The total initial state of the four systems is given by:

$$\rho_{AB} \otimes |0\rangle_{CD} \langle 0| \,. \tag{12}$$

The systems then evolve under a unitary interaction U, which acts on the entire composite system:

$$\rho' = U(\rho_{AB} \otimes |0\rangle_{CD} \langle 0|)U^\dagger. \tag{13}$$

After the interaction, we discard systems A and D, which corresponds to taking the partial trace over those subsystems:

$$E(\rho) = \mathrm{Tr}_{AD}(\rho'). \tag{14}$$

Using a basis $\{|a\rangle\}$ for system A and a basis $\{|d\rangle\}$ for system D, we can express the partial trace as:

$$E(\rho) = \sum_{a,d} \langle a, d| \, U(\rho \otimes |0\rangle \langle 0|)U^\dagger \, |a, d\rangle \,. \tag{15}$$

Defining the operators:

$$E_k = \langle k| \, U \, |0\rangle \,, \tag{16}$$

where $|k\rangle$ labels the basis vectors of systems A and D, we can rewrite the expression as:

$$E(\rho) = \sum_k E_k \rho E_k^\dagger. \tag{17}$$

Finally, since U is unitary, it preserves inner products, which implies:

$$\sum_k E_k^\dagger E_k = I. \tag{18}$$

Thus, we have shown that the map $E(\rho)$ satisfies the operator-sum representation with Kraus operators E_k, completing the proof.

Exercise 8.10 (Nielsen and Chuang): Using the Freedom in Operator-Sum Representation, all quantum operations E on a system of Hilbert space dimension d can be generated by an operator-sum representation containing at most d^2 elements:

$$E(\rho) = \sum_{j=1}^{M} E_j \rho E_j^\dagger, \tag{19}$$

where $1 \leq M \leq d^2$. Let $\{E_j\}$ be a set of operation elements for E. Define a matrix:

$$W_{jk} \equiv \mathrm{Tr}(E_j^\dagger E_k). \tag{20}$$

Show that the matrix W is Hermitian and of rank at most d^2, and thus there exists a unitary matrix u such that $uWu^\dagger$ is diagonal with at most d^2 nonzero entries. Use u to define a new set of at most d^2 nonzero operation elements $\{F_j\}$ for E.

Solution: We begin by defining the matrix W whose elements are given by:

$$W_{jk} = \mathrm{Tr}(E_j^\dagger E_k). \tag{21}$$

Since the trace inner product satisfies $\mathrm{Tr}(A^\dagger B) = \mathrm{Tr}(B^\dagger A)$, it follows that:

$$W_{jk} = \mathrm{Tr}(E_j^\dagger E_k) = \overline{W_{kj}}, \tag{22}$$

which implies that W is a Hermitian matrix.

The rank of W is determined by the number of linearly independent operators in the set $\{E_j\}$. Since the operators act on a Hilbert space of dimension d, they can be represented as $d \times d$ matrices. Consequently, the space of all possible operators has dimension at most d^2. Therefore, the rank of W is at most d^2.

By the spectral theorem for Hermitian matrices, there exists a unitary matrix u such that $uWu^\dagger$ is diagonal. Explicitly, we can write:

$$uWu^\dagger = \Lambda, \tag{23}$$

where Λ is a diagonal matrix with at most d^2 nonzero entries, given that $\mathrm{rank}(W) \leq d^2$.

We now define a new set of operation elements $\{F_j\}$ by:

$$F_j = \sum_k u_{jk} E_k. \tag{24}$$

Substituting this into the operator-sum representation:

$$E(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_{j,k} u_{jk} E_k \rho \sum_m u_{jm}^* E_m^\dagger. \tag{25}$$

Rearranging sums, we obtain:

$$E(\rho) = \sum_j F_j \rho F_j^\dagger. \tag{26}$$

Since u is unitary, the new set $\{F_j\}$ also satisfies the completeness relation:

$$\sum_j F_j^\dagger F_j = \sum_{j,k,m} u_{jk}^* E_k^\dagger u_{jm} E_m = \sum_{k,m} \delta_{km} E_k^\dagger E_m = \sum_k E_k^\dagger E_k = I. \tag{27}$$

Thus, we have constructed a new operator-sum representation for E with at most d^2 nonzero terms, completing the proof.

Exercise 8.11 (Nielsen and Chuang): Suppose E is a quantum operation mapping a d-dimensional input space to a d'-dimensional output space. Show that E can be described using a set of at most dd' operation elements $\{E_k\}$.

Solution: A quantum operation E can be described by an operator-sum representation:

$$E(\rho) = \sum_k E_k \rho E_k^\dagger, \tag{28}$$

where $\{E_k\}$ are the operation elements. The goal is to show that we can always find a set with at most dd' elements.

The operators E_k map a d-dimensional Hilbert space to a d'-dimensional Hilbert space, meaning that each E_k is a $d' \times d$ matrix. The space of all possible linear operators acting between these two Hilbert spaces has dimension dd', since an arbitrary matrix of size $d' \times d$ has dd' independent components.

Consider an arbitrary set of operation elements $\{E_j\}$. Let us define the matrix:

$$W_{jk} = \mathrm{Tr}(E_j^\dagger E_k). \tag{29}$$

This matrix W is Hermitian, and its rank is at most dd', as it is defined by at most dd' linearly independent operators in the space of $d' \times d$ matrices.

By the spectral theorem, there exists a unitary matrix u such that:

$$uWu^\dagger = \Lambda, \tag{30}$$

where Λ is diagonal with at most dd' nonzero entries. Defining a new set of operation elements:

$$F_j = \sum_k u_{jk} E_k, \tag{31}$$

we obtain an equivalent operator-sum representation:

$$E(\rho) = \sum_j F_j \rho F_j^\dagger. \tag{32}$$

Since W has rank at most dd', there are at most dd' nonzero elements in the new set $\{F_j\}$. Thus, we conclude that any quantum operation E mapping a d-dimensional input space to a d'-dimensional output space can always be represented using at most dd' operation elements, completing the proof.

4. **PROBLEM 4:**
Consider a qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ Suppose phase noise acts on this state, dephasizing the qubit. This action can be described as a unitary action on the qubit such that the rotations $R_Z(\theta)$ act on the qubit according to uniform distribution over θ. Obtain the resulting density matrix. Further, suppose that the longitudinal and tranverse relaxation times of the qubit are T_1 and T_2 , respectively. Obtain the final density matrix as a function of all the given parameters, and physically interpret your results geometrically over the Bloch sphere. How do you generalize this setup for a composite system when the qubits are in (a) product state and (b) entangled state? You need to bring in the relaxation time parameters to the composite system carefully within the density matrix formulation. This gives you an idea what happens when the qubits are not identical and what it takes to control relaxation times.

Solution:
Consider a qubit in the state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{33}$$

When phase noise acts on the qubit, the unitary operation is given by a random rotation about the z-axis:

$$R_Z(\theta) = e^{-i\theta Z/2}, \tag{34}$$

where θ is uniformly distributed over $[0, 2\pi]$. The initial density matrix of the qubit is:

$$\rho = |\psi\rangle\langle\psi| = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}. \tag{35}$$

Applying phase noise, we compute:

$$\rho' = \int_0^{2\pi} R_Z(\theta)\rho R_Z^\dagger(\theta)\frac{d\theta}{2\pi}. \tag{36}$$

Since $R_Z(\theta)$ applies a phase shift, the off-diagonal elements accumulate a phase factor:

$$\rho' = \int_0^{2\pi} \begin{bmatrix} |\alpha|^2 & \alpha\beta^* e^{-i\theta} \\ \alpha^*\beta e^{i\theta} & |\beta|^2 \end{bmatrix} \frac{d\theta}{2\pi}. \tag{37}$$

The integral eliminates off-diagonal terms, yielding:

$$\rho' = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}. \tag{38}$$

Thus, phase noise leads to dephasing, removing quantum coherence.

Effect of longitudinal (T_1) and transverse (T_2) relaxation times. The initial density matrix is

$$\rho(0) = \begin{bmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{bmatrix}$$

i) Longitudinal relaxation (T_1) - energy decay

- This accounts for the relaxation of the excited state $|1\rangle$ to the ground state $|0\rangle$ - The probability of $|1\rangle$ decays exponentially with 't':

$$p_1(t) = |b|^2 e^{-t/T_1}$$

Total probability is equal to 1, we have

$$p_0(t) = |a|^2 + |b|^2(1 - e^{-t/T_1})$$

ii) Transverse relaxation (T_2) - Decoherence - This accounts for the decay of the off-diagonal terms in $S(0)$, which represent quantum coherence - The coherence terms decay as

$$\rho_{01}(t) = ab^* e^{-t/T_2}$$
$$\rho_{10}(t) = a^* b e^{-t/T_2}$$

the density matrix at time 't' is

$$\rho(t) = \begin{bmatrix} |a|^2 + |b|^2(1 - e^{-t/T_1}) & ab^* e^{-t/T_2} \\ a^* b e^{-t/T_2} & |b|^2 e^{-t/T_1} \end{bmatrix}$$

$$\Rightarrow \rho(t) = \begin{bmatrix} 1 - |b|^2 e^{-t/T_1} & ab^* e^{-t/T_2} \\ a^* b e^{-t/T_2} & |b|^2 e^{-t/T_1} \end{bmatrix}$$

The term $\rho_{00}(t)$ can be rewritten as

$$\frac{1 + (|a|^2 - |b|^2) e^{-t/T_1} + (1 - e^{-t/T_1})}{2}$$

and $\rho_{11}(t)$ can be rewritten as

$$\frac{1 - (|a|^2 - |b|^2) e^{-t/T_1} - (1 - e^{-t/T_1})}{2}$$

Geometric interpretation on the Bloch sphere:

Comparing $\rho(t)$ with

$$\frac{1}{2}\left(I + r_x X + r_y Y + r_z Z\right)$$

we obtain

$$r_x = \mathrm{Re}\left(a^* b\right) e^{-t/T_2}$$
$$r_y = \mathrm{Im}\left(a^* b\right) e^{-t/T_2}$$
$$r_z = \left(|a|^2 - |b|^2\right) e^{-t/T_1} + \left(1 - e^{-t/T_1}\right)$$

We can draw the following conclusions

(a) The off-diagonal terms $\rho_{01}(t)$ and $\rho_{10}(t)$ shrink as e^{-t/T_2}, meaning the Bloch vector components r_x and r_y decay towards zero.

(b) This corresponds to a shrinking of the Bloch vector in the xy-plane, reducing the phase coherence of the qubit.

(c) The diagonal terms shift towards 1, resulting in $r_z \to 1$.

(d) This corresponds to the Bloch vector moving vertically towards the north pole (ground state $|0\rangle$).

In conclusion,

(i) T_1 governs the relaxation towards the ground state, reducing the probability amplitude of $|1\rangle$.

(ii) T_2 governs the loss of coherence, causing the qubit to become classically mixed rather than maintaining superposition.

(iii) This captures the irreversible loss of quantum information due to decoherence and energy dissipation.

Generalisation to composite systems

For Product state Consider the initial state of a product state

$$\rho(0) = \rho_A(0) \otimes \rho_B(0)$$

where,

$$\rho_A(0) = \begin{bmatrix} |a_A|^2 & a_A b_A^* \\ a_A^* b_A & |b_A|^2 \end{bmatrix} \quad \rho_B(0) = \begin{bmatrix} |a_B|^2 & a_B b_B^* \\ a_B^* b_B & |b_B|^2 \end{bmatrix}$$

$$\rho(0) = \begin{bmatrix} |a_A|^2|a_B|^2 & |a_A|^2 a_B b_B^* & a_A b_A^* |a_B|^2 & a_A b_A^* a_B b_B^* \\ |a_A|^2 a_B^* b_B & |a_A|^2 |b_B|^2 & a_A b_A^* a_B^* b_B & a_A b_A^* |b_B|^2 \\ a_A^* b_A |a_B|^2 & a_A^* b_A a_B b_B^* & |b_A|^2 |a_B|^2 & |b_A|^2 a_B b_B^* \\ a_A^* b_A a_B^* b_B & a_A^* b_A |b_B|^2 & |b_A|^2 a_B^* b_B & |b_A|^2 |b_B|^2 \end{bmatrix}$$

Similar to the previous case, we obtain

$$\rho_i(t) = \begin{bmatrix} |a_i|^2 + |b_i|^2(1 - e^{-t/T_{1i}}) & a_i b_i^* e^{-t/T_{2i}} \\ a_i^* b_i e^{-t/T_{2i}} & |b_i|^2 e^{-t/T_{1i}} \end{bmatrix} \quad i \in A, B$$

$$\rho(t) = \rho_A(t) \otimes \rho_B(t)$$

We have the following diagonal terms of $\rho(t)$:

$$\rho_{00}(t) = \left[|a_A|^2 + |b_A|^2(1 - e^{-t/T_{1A}})\right] \left[|a_B|^2 + |b_B|^2(1 - e^{-t/T_{1B}})\right]$$

$$\rho_{11}(t) = \left[|a_A|^2 + |b_A|^2(1 - e^{-t/T_{1A}})\right] \left[|b_B|^2 e^{-t/T_{1B}}\right]$$

$$\rho_{22}(t) = \left[|b_A|^2 e^{-t/T_{1A}}\right] \left[|a_A|^2 + |b_A|^2(1 - e^{-t/T_{1A}})\right]$$

$$\rho_{33}(t) = \left[|b_A|^2 e^{-t/T_{1A}}\right]\left[|b_B|^2 e^{-t/T_{1B}}\right]$$

And the off-diagonal terms are of the form:

$$\rho_{ij}(t) = (K_1 e^{-t/T_{m_1,n_1}})(K_2 e^{-t/T_{m_2,n_2}})$$

for $i \neq j$, $m_1, m_2 \in \{1, 2\}$, $n_1, n_2 \in \{A, B\}$.

Interpretation of the density matrix evolution

- The diagonal elements represent probabilities of measuring each computational basis state $|00\rangle, |01\rangle, |10\rangle, |11\rangle$
  - They increase towards $|00\rangle$ over time as relaxation drives each qubit to the ground state.
  - The rate depends on T_{1A} and T_{1B}, i.e., if the qubits are not identical, they relax at different speeds.
- The off-diagonal elements represent coherence.
  - These decay exponentially at rates T_{2A} and T_{2B}, i.e., the degree of quantum interference decreases over time.
- If both $T_{1A}, T_{1B} \to \infty$, the qubits stay in their initial state.

Bloch vector interpretation

- Each qubit independently shrinks towards the north pole $|0\rangle$ at rates T_{1A}, T_{1B}.
- The Bloch vector lengths decrease, reflecting the loss of coherence at rates T_{2A}, T_{2B}.
- Unequal relaxation times distort the Bloch sphere asymmetrically.

Entangled state

Consider the initial state $|\Psi(0)\rangle$ defined by

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{(Bell state)}$$

The corresponding density matrix is

$$\rho(0) = |\Psi(0)\rangle\langle\Psi(0)| = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

This state exhibits maximal quantum coherence due to the off-diagonal terms. Assume the evolution of the following form:

$$\rho(t) = \mathcal{E}_A \otimes \mathcal{E}_B(\rho(0))$$

where

$$\mathcal{E}(\rho_i(t)) = \begin{bmatrix} |a_i|^2 + |b_i|^2(1 - e^{-t/T_{1i}}) & a_i b_i^* e^{-t/T_{2i}} \\ a_i^* b_i e^{-t/T_{2i}} & |b_i|^2 e^{-t/T_{1i}} \end{bmatrix}, \quad i \in \{A, B\}$$

as derived in the previous case.

$$\rho(t) = \begin{bmatrix} \frac{1+e^{-t/T_{1A}}e^{-t/T_{1B}}}{2} & 0 & 0 & e^{-t/T_{2A}}e^{-t/T_{2B}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-t/T_{2A}}e^{-t/T_{2B}} & 0 & 0 & \frac{1-e^{-t/T_{1A}}e^{-t/T_{1B}}}{2} \end{bmatrix}$$

Interpretation

- The probability amplitude of $|00\rangle$ increases over time, i.e., the system relaxes to $|00\rangle$.
- The probability amplitude of $|11\rangle$ decays at a rate of $e^{-t/T_{1A}}e^{-t/T_{1B}}$.
- If $T_{1A} \neq T_{1B}$, the relaxation is asymmetric, meaning one qubit may relax faster than the other.

Coherence

- The coherence term $e^{-t/T_{2A}}e^{-t/T_{2B}}$ decays exponentially.
- Faster dephasing (small T_2) destroys entanglement quickly.
- If $T_{2A} \neq T_{2B}$, one qubit loses coherence faster, destroying the entanglement earlier.

Interpretation on Bloch Sphere

- The system shrinks towards the north pole ($|00\rangle$).
- The Bloch vector length decreases, representing loss of quantum coherence.
- The degree of entanglement decreases over time, transitioning into a mixed state.

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References

[1] R. A. Bertlmann and N. Friis, *Modern Quantum Theory: From Quantum Mechanics to Entanglement and Quantum Information*, Oxford University Press, 2023.