Quantum Information Theory (E2-270) (Spring 2025) **HW1**
Instructor: Prof. Shayan Srinivasa Garani

1. **PROBLEM 1:** Consider the standard quantum teleportation protocol for teleporting a quantum qubit state $|\psi\rangle$. The measurement outcomes in the Bell basis must be relayed to the receiver to reconstruct the quantum state. Suppose the measurement outcome is corrupted by noise which can be modeled using a binary symmetric channel with crossover probabiltiy p, what is the reconstruction fidelity at the output? Suggest a simple scheme to improve this reconstruction fidelity. Justify all your reasoning carefully, including the teleportation part.

Solution:
Quantum teleportation is a fundamental protocol in quantum information science, enabling the transfer of an arbitrary quantum state between two parties, Alice and Bob, using shared entanglement and classical communication. The standard teleportation protocol proceeds as follows. First, Alice and Bob share a maximally entangled Bell state, such as

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Alice then performs a Bell state measurement on her unknown qubit, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, along with her half of the entangled pair. The measurement outcome collapses her qubits into one of four Bell states:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \quad |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

Alice then transmits two classical bits encoding the measurement result to Bob. Upon receiving this information, Bob applies the appropriate Pauli correction I, X, Z, XZ to recover $|\psi\rangle$ perfectly.

However, in the classical communication channel may be noisy. A common model for noise is the **binary symmetric channel (BSC)**, where each transmitted bit has a probability p of being flipped. Given that Alice transmits two bits, the possible transmission scenarios and their probabilities are:

- Both bits transmitted correctly: probability $(1-p)^2$.
- One bit flipped: probability $2p(1-p)$.
- Both bits flipped: probability p^2.

If a bit flip occurs, Bob applies an incorrect Pauli operation, which degrades the fidelity of the reconstructed state. The fidelity F measures the closeness of the output state ρ_{out} to the original state $|\psi\rangle$, given by

$$F = \langle\psi| \rho_{\text{out}} |\psi\rangle .$$

For a noiseless channel ($p = 0$), the fidelity is perfect, $F = 1$. When noise is present, we compute the expected fidelity as follows:

$$F = (1-p)^2 \cdot 1 + 2p(1-p) \cdot 0 + p^2 \cdot 0 = 1 - 2p + p^2.$$

To mitigate the effect of noise, we employ **error correction techniques**. A simple approach is **repetition coding**, where each bit is sent three times (e.g., $0 \to 000$, $1 \to 111$). Bob then decodes each bit using majority voting. The probability of correctly decoding a bit is

$$P_{\text{correct}} = (1-p)^3 + 3p(1-p)^2.$$

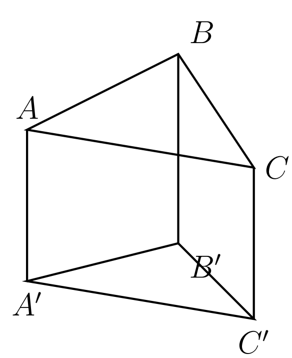
Thus, the probability of correctly receiving both bits is P_{correct}^2, leading to an improved fidelity:

$$F_{\text{improved}} = \left((1-p)^3 + 3p(1-p)^2\right)^2.$$

For small p, repetition coding significantly enhances fidelity. For instance, if $p = 0.1$, the original fidelity is $F = 0.82$, while with repetition coding, $F_{\text{improved}} \approx 0.945$. In conclusion, while teleportation fidelity degrades under a noisy classical channel, error correction strategies such as repetition coding can effectively mitigate its effects and improve the reliability of quantum teleportation.

2. **PROBLEM 2:** Consider a triangular prism with vertices A, B, C on the top and the corresponding vertices A', B', C' at the bottom. A spider and an ant are initially sitting on vertices A and C', respectively. At each time step, both of them traverse only along an edge of the prism. The choice of an edge is equally likely from the starting vertex at any time step. At any time while on an edge, they do not reverse their directions. What is the expected number of steps taken before the spider and the ant meet? What is the entropy rate of this random walk process until the spider and the ant meet?

Solution:



Possible States

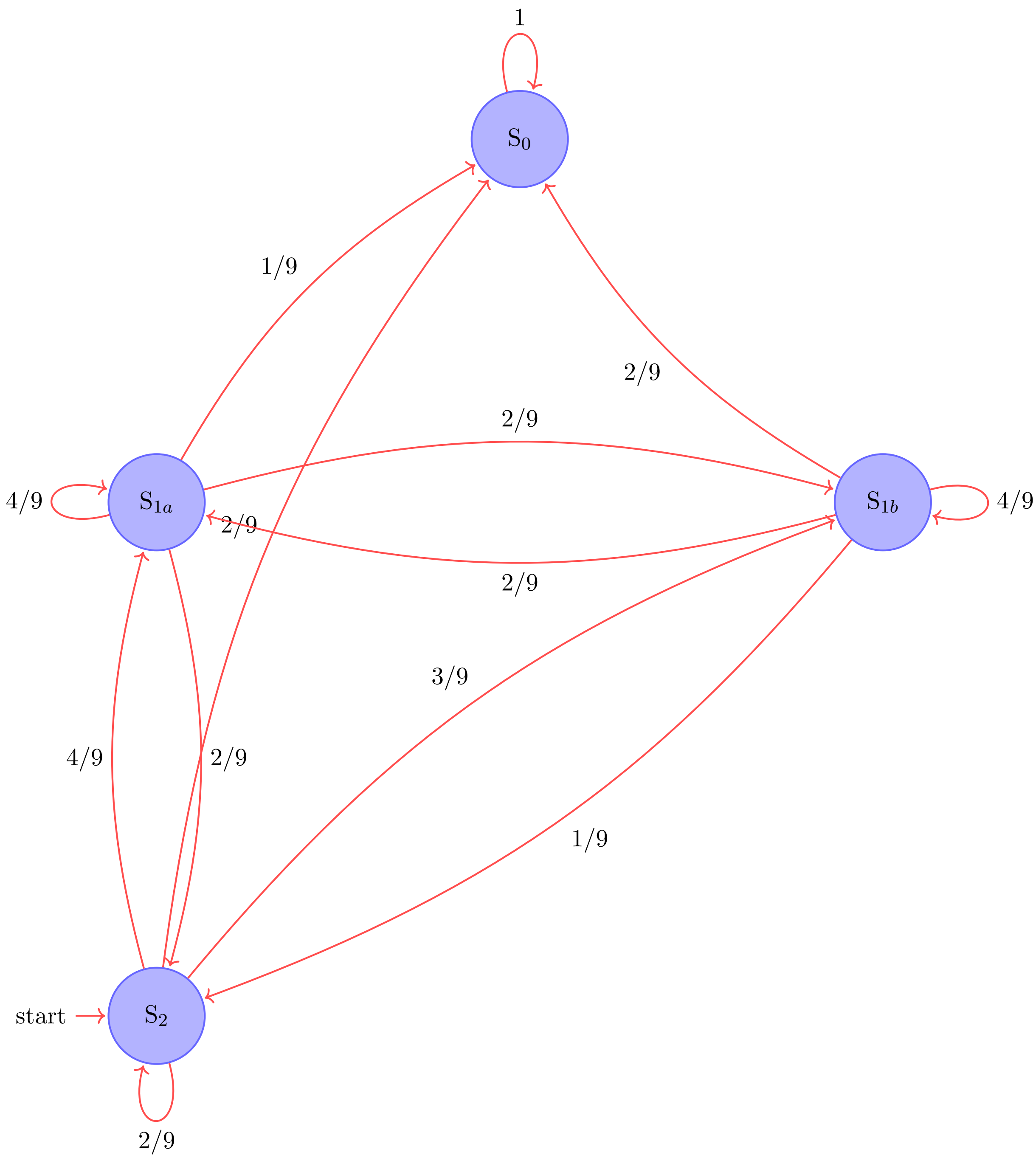
1. State 0: Spider and ant are at the same vertex (they meet). This can be considered to be an absorbing state.

2. State 1a: Spider and ant are on adjacent vertices (distance = 1) and on the same face.

3. State 1b: Spider and ant are on adjacent vertices (distance = 1) and on opposite faces.

4. State 2: Spider and ant are separated by 2 vertices (distance = 2).

These are the only possible states as the maximum number of edges by which two vertices are separated on a triangular prism is 2.



Transition Probabilities

From state 1, the spider and ant can move closer (to state 0), stay at the same distance (state 1), or move farther apart (state 2).

From state 2, the spider and the ant can move closer by 1 edge (state 1), stay at the same distance (state 2), or move closer by 2 edges (state 0).

Computation of Transition Probabilities

Let us compute the transition probabilities when the spider is at vertex A and ant at C, i.e., state 2. Since the transition process is symmetric, the state transition probabilities will be the same for all combinations of vertices representing state 2.

The spider from A can move to B, C, and A', i.e., it has 3 choices. Similarly, the ant at C' has three choices of actions. Therefore, the total number of possible states after one time step is $3 \times 3 = 9$.

Possible Transitions at State 2

1) Meet (state 0) a) Spider moves to A' and ant moves to A'. b) Spider moves to C and ant moves to C'. c) Probability of meeting after starting at state 2 is:

$$P_4[S_{t+1} = 0|S_t = 2] = \frac{2}{9}$$

2) Transition to state 1a: a) Spider moves to A' and ant moves to B'. b) Spider moves to B and ant moves to C'. c) Probability:

$$P_4[S_{t+1} = 1A|S_t = 2] = \frac{2}{9}$$

3) Transition to state 1b: a) Spider moves to B and ant moves to B'. c) Probability:

$$P_4[S_{t+1} = 1b|S_t = 2] = \frac{1}{9}$$

4) Transition to state 2: a) Spider moves to A' and ant moves to C'. b) Spider moves to B and ant moves to A'. c) Spider moves to C and ant moves to A'. d) Spider moves to C and ant moves to B'.

$$P_7[S_{t+1} = 2|S_t = 2] = \frac{4}{9}$$

Possible Transitions at State 1a

1. Meet (State 0) a) Spider moves to B and ant moves to B' and meet on an edge.

$$P_2[S_{t+1} = 0 | S_t = 1] = \frac{2}{9}$$

2. Transition to state 1a: a) Spider moves to A' and ant moves to C'. b) Spider moves to B' and ant moves to A'. c) Spider moves to C' and ant moves to A'.

$$P_2[S_{t+1} = 1a | S_t = 1a] = \frac{3}{9}$$

3. Transition to state 1b: a) Spider moves to A' and ant moves to A. b) Spider moves to C and ant moves to C'.

$$P_2[S_{t+1} = 1b | S_t = 1a] = \frac{2}{9}$$

4. Transition to state 2: a) Spider moves to B and ant moves to C'. b) Spider moves to A' and ant moves to B.

$$P_2[S_{t+1} = 2 | S_t = 1a] = \frac{2}{9}$$

Possible Transitions at State 1b

1. Meet (State 0) Spider and ant move to A.

$$P_2[S_{t+1} = 0 | S_t = 1b] = \frac{1}{9}$$

2. Transition to state 1a: a) Spider moves to B and ant moves to C. b) Spider moves to A and ant moves to C. c) Spider moves to C' and ant moves to A'. d) Spider moves to C' and ant moves to B'.

$$P_7[S_{t+1} = 1a | S_t = 1b] = \frac{4}{9}$$

3. Transition to state 1b: a) Spider moves to A and ant moves to A'. b) Spider moves to B and ant moves to B'.

$$P_6[S_{t+1} = 1b | S_t = 1b] = \frac{2}{9}$$

4. Transition to state 2: a) Spider moves to A and ant moves to B'. b) Spider moves to B and ant moves to A'.

$$P_3[S_{t+1} = 2 | S_t = 1b] = \frac{2}{9}$$

The complete transition matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/9 & 3/9 & 2/9 & 2/9 \\ 1/9 & 4/9 & 2/9 & 2/9 \\ 2/9 & 2/9 & 1/9 & 4/9 \end{bmatrix} \tag{1}$$

Solving for Expected Number of Steps

Let E_i be the expected number of steps for the spider and the ant to meet starting from state i, where $i \in \{0, 1a, 1b, 2\}$.

1. For state 0: $E_0 = 0$
2. For state $1a$:

$$\begin{aligned} E_{1a} = & P(S_{t+1} = 0 | S_t = 1a)(1 + E_0) + P(S_{t+1} = 1a | S_t = 1a)(1 + E_{1a}) \\ & + P(S_{t+1} = 1b | S_t = 1a)(1 + E_{1b}) + P(S_{t+1} = 2 | S_t = 1a)(1 + E_2). \end{aligned}$$

$$E_{1a} = \frac{2}{9}(0+1) + \frac{3}{9}(1 + E_{1a}) + \frac{2}{9}(1 + E_{1b}) + \frac{2}{9}(1 + E_2) \quad (1)$$

3. For state $1b$:

$$\begin{aligned} E_{1b} = & P(S_{t+1} = 0 | S_t = 1b)(1 + E_0) + P(S_{t+1} = 1a | S_t = 1b)(1 + E_{1a}) \\ & + P(S_{t+1} = 1b | S_t = 1b)(1 + E_{1b}) + P(S_{t+1} = 2 | S_t = 2)(1 + E_2). \end{aligned}$$

$$E_{1b} = \frac{1}{9}(1 + E_0)\frac{4}{9}(1 + E_{1a}) + \frac{2}{9}(1 + E_{1b}) + \frac{2}{9}(1 + E_2) \quad (2)$$

4. For state 2:

$$\begin{aligned} E_2 = & P(S_{t+1} = 0 | S_t = 2)(1 + E_0) + P(S_{t+1} = 1a | S_t = 2)(1 + E_{1a}) \\ & + P(S_{t+1} = 1b | S_t = 2)(1 + E_{1b}) + P(S_{t+1} = 2 | S_t = 2)(1 + E_2). \end{aligned}$$

$$E_2 = \frac{2}{9} + \frac{2}{9}(1 + E_{1a}) + \frac{1}{9}(1 + E_{1b}) + \frac{4}{9}(1 + E_2) \quad (3)$$

Solving (1), (2), and (3) will get us a linear equation

$$\begin{bmatrix} 6/9 & -2/9 & -2/9 \\ -4/9 & 7/9 & -2/9 \\ -2/9 & -1/9 & 5/9 \end{bmatrix} \begin{bmatrix} E_{1a} \\ E_{1b} \\ E_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{2}$$

$E_{1a} = 4.973$, $E_{1b} = 5.526$ and $E_2 = 4.8947$.

Entropy Rate of the Given Problem

We are given the transition matrix P of an absorbing Markov chain:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/9 & 3/9 & 2/9 & 2/9 \\ 1/9 & 4/9 & 2/9 & 2/9 \\ 2/9 & 2/9 & 1/9 & 4/9 \end{bmatrix} \tag{3}$$

This matrix consists of: - An absorbing state S_1 (first row: once entered, it cannot be left). - Three transient states (S_2, S_3, S_4).

For an absorbing Markov chain, we write P in the standard form:

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} \tag{4}$$

where: - Q is the transition submatrix among transient states. - R describes transitions from transient to absorbing states.

From the given matrix P, we extract:

$$P' = \begin{bmatrix} 3/9 & 2/9 & 2/9 \\ 4/9 & 2/9 & 2/9 \\ 2/9 & 1/9 & 4/9 \end{bmatrix} \tag{5}$$

The fundamental matrix N is given by:

$$N = (I - Q)^{-1} \tag{6}$$

where I is the identity matrix of the same size as Q.

First, compute $I - Q$:

$$I - Q = \begin{bmatrix} 1 - \frac{3}{9} & -\frac{2}{9} & -\frac{2}{9} \\ -\frac{4}{9} & 1 - \frac{2}{9} & -\frac{2}{9} \\ -\frac{2}{9} & -\frac{1}{9} & 1 - \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{6}{9} & -\frac{2}{9} & -\frac{2}{9} \\ -\frac{4}{9} & \frac{7}{9} & -\frac{2}{9} \\ -\frac{2}{9} & -\frac{1}{9} & \frac{5}{9} \end{bmatrix} \tag{7}$$

Now, inverting this matrix gives:

$$N = \begin{bmatrix} 2.6 & 0.94 & 1.42 \\ 1.89 & 2.05 & 1.57 \\ 1.42 & 0.79 & 2.68 \end{bmatrix} \tag{8}$$

Each element N_{ij} represents the expected number of times the transient state j is visited, given that the chain started in state i. Therefore, sum the columns of the matrix N will get the answer as 4.974, 5.526, and 4.895 when initial states are S_2, S_3 and S_4. The final answer is 4.895.

The expected state occupancy distribution before absorption is given by:

$$\pi_j = \frac{1}{\sum_{i,j} N_{i,j}} \sum_i N_{i,j} \tag{9}$$

where $\sum_{i,j} N_{i,j}$ is the sum of all elements in N.
Computing $\sum_{i,j} N_{i,j}$:

$$\sum_{i,j} N_{i,j} = 15.4 \tag{10}$$

Summing over rows, we get the steady-state occupancy before absorption:

$$\pi = \begin{bmatrix} 0.3845 & 0.2461 & 0.3691 \end{bmatrix} \tag{11}$$

The entropy rate of the Markov chain before absorption is:

$$H_{\text{rate}} = -\sum_i \pi_i \sum_j Q_{ij} \log_2 Q_{ij} \tag{12}$$

Computing $Q_{ij} \log_2 Q_{ij}$:

$$Q \log_2 Q = \begin{bmatrix} \frac{3}{9}\log_2\frac{3}{9} & \frac{2}{9}\log_2\frac{2}{9} & \frac{2}{9}\log_2\frac{2}{9} \\ \frac{4}{9}\log_2\frac{4}{9} & \frac{2}{9}\log_2\frac{2}{9} & \frac{2}{9}\log_2\frac{2}{9} \\ \frac{2}{9}\log_2\frac{2}{9} & \frac{1}{9}\log_2\frac{1}{9} & \frac{4}{9}\log_2\frac{4}{9} \end{bmatrix} \tag{13}$$

Summing over all elements:

$$H_{\text{rate}} \approx 1.44 \tag{14}$$

The entropy rate before absorption is:

$$H_{\text{rate}} \approx 1.44 \text{ bits/step} \tag{15}$$

This quantifies the average uncertainty per step in the transient states before the system is absorbed.

3. **PROBLEM 3:** A rudimentary channel model for reading the charge from M-ary-based flash memory cells can be described using an extended version of the discrete binary symmetric channel extended for the M-ary inputs.

(1) Assuming that the crossover probabilities are the same across the symbols, derive an expression for the channel capacity of the model.

Solution:

The channel is an M-ary symmetric channel where

- Each input symbol is correctly received with probability $1 - p$.
- Each input symbol is received as one of the $M - 1$ other symbols with probability $\frac{p}{M-1}$.

The channel capacity is given by:

$$C = \max_{P(X)} I(X;Y),$$

where $I(X;Y)$ is the mutual information:

$$I(X;Y) = H(Y) - H(Y|X).$$

For a uniform input distribution, the output entropy is:

$$H(Y) = \log M.$$

The conditional entropy is:

$$H(Y|X) = -(1-p)\log(1-p) - p\log\frac{p}{M-1}.$$

Thus, the channel capacity is:

$$C = \log M + (1-p)\log(1-p) + p\log\frac{p}{M-1}.$$

(2) Suppose we have a bad flash memory device due to manufacturing, where the crossover probabilities are time-varying. Let $\{Y_i\}_{i=1}^n$ be the random variables sensed at the analog-to-digital converter (ADC) output of this M-ary cell, corresponding to the inputs $\{X_i\}_{i=1}^n$, assumed to be conditionally independent. Let the conditional distribution be given by:

$$p(y|x) = \prod_{i=1}^{n} p_i(y_i|x_i)$$

for n reads. Obtain $\max_{p(x)} I(X;Y)$ using the setup from the previous part.

Solution:

The mutual information is given by:

$$I(X;Y) = H(Y) - H(Y|X).$$

Since the channel is memoryless, the conditional entropy decomposes as:

$$H(Y|X) = \sum_{i=1}^{n} H(Y_i|X_i).$$

For each channel use,

$$H(Y_i|X_i) = -(1-p_i)\log(1-p_i) - p_i \log \frac{p_i}{M-1}.$$

The output entropy is:

$$H(Y) = \sum_{i=1}^{n} \log M = n \log M.$$

Thus, the channel capacity is:

$$C = \sum_{i=1}^{n} \left[\log M + (1-p_i)\log(1-p_i) + p_i \log \frac{p_i}{M-1} \right].$$

This can be interpreted as an M-ary physical channel, where n number of M-dits are transmitted or stored using an n-Mdit classical code to protect against errors introduced by the channel with capacity C.

4. **PROBLEM 4:**
(1) Obtain the stationary states of the single-qubit Hamiltonian for the following cases: $H = \hbar\omega Z$ and $H = \hbar\omega H$, where Z and H are the usual phase flip and Hadamard operators, respectively. Also, obtain the evolution of the stationary states over time and provide a physical interpretation of the results.

Solution:
Case 1: $H = \hbar\omega Z$ The Pauli-Z matrix is given by:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues and eigenvectors of Z are:

$$Z|0\rangle = |0\rangle, \quad Z|1\rangle = -|1\rangle.$$

Since $H = \hbar\omega Z$, the stationary states are $|0\rangle$ and $|1\rangle$, with energies:

$$E_0 = \hbar\omega, \quad E_1 = -\hbar\omega.$$

Time evolution of a stationary state $|\psi(0)\rangle$ follows:

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle .$$

Thus,

$$|0(t)\rangle = e^{-i\omega t} |0\rangle , \quad |1(t)\rangle = e^{i\omega t} |1\rangle .$$

Physically, this implies the rotation of the Block sphere around the polar axis with an angle ω.

Case 2: $H = \hbar\omega H$ The Hadamard matrix is:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} .$$

Finding its eigenvalues:

$$\det(H - \lambda I) = \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = 0.$$

Solving, we get eigenvalues ± 1 with corresponding eigenvectors:

$$|H_-\rangle = 0.3827 |0\rangle - 0.9239 |1\rangle , \quad |H_+\rangle = -0.9239 |0\rangle - 0.3827 |1\rangle .$$

The time evolution follows similarly:

$$|H_+(t)\rangle = e^{-i\omega t} |H_+\rangle , \quad |H_-(t)\rangle = e^{i\omega t} |H_-\rangle .$$

Physically, this implies the rotation of the Block sphere around the Hadamard axis (quantum Fourier transform axis) with an angle ω.

(2) Suppose we prepare an ensemble of Bell states:

$$\left\{ \left(\frac{1}{2}, |\Phi^+\rangle\right), \left(\frac{1}{4}, |\Phi^-\rangle\right), \left(\frac{1}{8}, |\Psi^+\rangle\right), \left(\frac{1}{8}, |\Psi^-\rangle\right) \right\} .$$

Compute the expectation values of the observables:

(a) XX

(b) XZ

Solution:
The Bell states are:

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\ |\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned}$$

The expectation value of an observable O over the ensemble is:

$$\mathbb{E}[O] = \sum_i p_i \langle \psi_i | O | \psi_i \rangle.$$

For XX:

$$\langle \Phi^+ | XX | \Phi^+ \rangle = 1, \quad \langle \Phi^- | XX | \Phi^- \rangle = -1,$$
$$\langle \Psi^+ | XX | \Psi^+ \rangle = 1, \quad \langle \Psi^- | XX | \Psi^- \rangle = -1.$$

Thus,

$$\mathbb{E}[XX] = \left(\frac{1}{2} + \frac{1}{8} \right)(1) + \left(\frac{1}{4} + \frac{1}{8} \right)(-1) = \frac{5}{8} - \frac{3}{8} = \frac{1}{4}.$$

For XZ:

$$\langle \Phi^+ | XZ | \Phi^+ \rangle = 0, \quad \langle \Phi^- | XZ | \Phi^- \rangle = 0,$$
$$\langle \Psi^+ | XZ | \Psi^+ \rangle = 0, \quad \langle \Psi^- | XZ | \Psi^- \rangle = 0.$$

Thus,

$$\mathbb{E}[XZ] = 0.$$

Acknowledgement

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