Indian Institute of Science

Mathematical Methods and Techniques in Signal Processing

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Home Work #3, Spring 2024

Late submission policy: Points scored = Correct points scored $\times e^{-d}$, d = # days late

Assigned date: Apr. 5^{th}, 2024 **Due date:** Apr. 17^{th}, 2024, 11:59 pm.

Problem 1. Derive a wavelet decomposition for a p-adic Haar basis. Show the link to analysis and synthesis filter banks. Obtain the time-frequency uncertainty relationship at each scale. What do you conclude?

Problem 2. You are given a chessboard image, i.e., with black and white squares that could be represented with alternating zeros and ones. Suppose you would like to represent this image using (a) a 2D Haar basis (b) 2D DFT, what would that look like? If you were to develop a compression engine and recover this data perfectly using (a) and (b), what would be the compression rates? What can you comment on the efficiency against the best compression ratio possible for this image?

Problem 3. Suppose $1 < p < \infty$ and $f \in L^p[a,b]$. Then define the Fourier series of a function $f(x) \in L^2[a,b]$ with respect to a orthogonal basis other than $\{\sin(\frac{2n\pi x}{b-a}), \cos(\frac{2n\pi x}{b-a})\}_{n=0}^{\infty}$. Prove or disprove that the Fourier series of a function $f(x) \in L^p[a,b]$ with respect to the orthogonal basis $\{\sin(\frac{2n\pi x}{b-a}), \cos(\frac{2n\pi x}{b-a})\}_{n=0}^{\infty}$ converges in the norm of $L^p[a,b]$.

Problem 4. Let $2 \leq p < q < \infty$. Prove or disprove that convergence of the Fourier series in the norm of $L^p[a,b]$ implies convergence in the norm of $L^q[a,b]$. If there is no relation between convergences in the norms of $L^p[a,b]$ and $L^q[a,b]$ then justify your answer.

Problem 5. Suppose that (f_n) is a sequence of differentiable functions $f_n : (a,b) \longrightarrow \mathbb{R}$ such that $f_n \longrightarrow f$ pointwise and $f_n' \longrightarrow g$ uniformly for some $f, g : (a,b) \longrightarrow \mathbb{R}$. Then f is differentiable on (a,b) and $f' = g$.
(a) Prove or disprove: What will happen if we lose the condition of uniform convergence of f_n'?
(b) Prove or disprove: What can we say about differentiability, if (f_n') is assumed to be continuous?

Problem 6. If $\langle f_n \rangle$ is a sequence of measurable functions that converge to a real-valued function f almost everywhere on a measurable set E of finite measure, then given $\eta > 0$, there is a subset $A \subset E$ with $mA < \eta$ such that f_n converges to f uniformly on $E \setminus A$.
{**Hint**: You may utilize Proposition 24, p. 74 of the book "Real Analysis, 3rd Ed. by H.L. Royden"}

Problem 7. Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Construct a sequence $\langle f_n \rangle$ of nonnegative, Riemann integrable functions such that f_n increases monotonically to f. Explain a process for constructing such sequence.
What does this imply about changing the order of integration and the limiting process. Motivate your response through appropriate justification.

Problem 8. Consider the n-dimensional vector space $V := \mathbb{R}^n$ over $\mathbb{R}$. Let $u \in V$ with $|Support(u)| = 1$ Define $W := Span(u)$, a vector subspace of V. Let α, β be two positive real numbers such that $\beta = \sqrt{\alpha}$. Define $\tilde{W}_\alpha := \{x_i : x_i \leq \alpha \; \forall \, i \in \text{Support}(u), x \in W\}$ and $\tilde{\tilde{W}}_\beta := \{y + \beta : y \in \tilde{W}_\alpha\}$.
What can you say about the measurability of the sets $\tilde{W}_\alpha$ and $\tilde{\tilde{W}}_\beta$? Justify your reasoning!