Indian Institute of Science

E2: 270 Quantum Information Theory

Instructor: Shayan Srinivasa Garani

MidTerm Exam #2, Spring 2023

Name and SR.No:

Instructions:

- This is an in-class exam for 2 hours.
- There are three main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- If required, make any necessary assumptions. However, state them clearly.
- There is absolutely no collaboration except referring to your two crib sheets.
- Assigned on Nov. 13^{th}, 2023 at 18:30 hrs.
- Do not panic, do not cheat, good luck!

Question No.	Points scored
1	
2	
3	
Total points	

PROBLEM 1: This problem has 3 parts:

(1) Let $|\psi\rangle_{ABC}$ be a tripartite state of the form (a) $\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$ and (b) $\frac{1}{\sqrt{3}}(|100\rangle+|010\rangle+|001\rangle)$). What is the entropy of the states in (a) and (b) initially and after tracing out system A? Comment on the purity of the bipartite quantum states after tracing out system A. (5 pts.)

(2) Prove the monotonicity of trace distance and fidelity under a quantum channel action $\mathcal{N}$. (8 pts.)

(3) Prove that the entanglement fidelity is convex in the input state under the quantum channel action $\mathcal{N}$. (7 pts.)

PROBLEM 2: This problem has 3 parts:

(1) In exam 1, you were asked to evaluate the Shannon entropy of a quantum ensemble in spectral form. In this exam problem, you will actually attempt to prove this. Given an ensemble $\{p_X(x), |\psi_x\rangle\}$, prove that the von Neumann entropy $H(\rho) \leq H(X)$, where $H(X)$ is the Shannon entropy. (7 pts.)

(2) A student attempted to purify a noisy quantum state in the *same* Hilbert space and claimed that it was always possible to achieve full purity by attempting to do quantum operations. Justify if this is possible or not mathematically. Illustrate through a quantum circuit how a maximally mixed state could be converted to a pure Bell state $|\Phi^+\rangle$. Provide a geometric feel of the problem. During the process of achieving purification, prove why the quantum entropy decreases. (8 pts.)

(3) Following up on the second part of the problem, suppose that the student started with a pure state $|\psi\rangle$ and landed up with two mixed states $\rho^{(1)}$ and $\rho^{(2)}$ during two consecutive attempts towards purifying the states over the same Hilbert space using an *arbitrary* random quantum operation.Justify if $F(\rho^{(1)}, |\psi\rangle) \leq F(\rho^{(2)}, |\psi\rangle) + ||\rho^{(1)} - \rho^{(2)}||$ holds true. (5 pts.)

PROBLEM 3: This problem has 2 parts:

(1) Prove that $I(A;B) \leq 2\min(\log(d_A), \log(d_B))$, where d_A and d_B are the dimensions of quantum systems A and B. (4 pts.)

(2) Consider the action of isometry $U^{A \to BE}$ on a tripartite state $|\psi\rangle^{SRA}$ to produce $\left|\phi^{SRBE}\right\rangle$. Show that $I(R;A)_{|\psi\rangle} + I(R;S)_{|\psi\rangle} = I(R;B)_{|\phi\rangle} + I(R;SE)_{|\phi\rangle}$ (6 pts.)