Indian Institute of Science

E2: 270 Quantum Information Theory

Instructor: Shayan Srinivasa Garani

MidTerm Exam, Spring 2023

Name and SR.No:

Instructions:

- This is an in-class exam for 2 hours.
- There are five main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- If required, make any necessary assumptions. However, state them clearly.
- There is absolutely no collaboration except referring to your crib sheets and homework solutions.
- Assigned on Oct. 4th, 2023 at 18:30 hrs.
- Do not panic, do not cheat, good luck!

Question No.	Points scored
1	
2	
3	
4	
5	
Total points	

PROBLEM 1: This problem has 2 parts:

(1) Assume that Alice is communicating to Bob using two non-orthogonal single qubit states. Eve is wiretapping this information *unitarily* using an ancilla to gain information into Alice's states. Prove that in order to distinguish the states at Eve's side, she must inevitably disturb one of the states. Hint: Set up the variables carefully as you need to prove this result. (5 pts.)

(2) Consider the quantum ensemble of non-orthogonal states $\left\{\frac{1}{4}\left|0\right\rangle, \frac{3}{4}\left|+\right\rangle\right\}$. What is the Shannon entropy for this source? Obtain the density matrix for this ensemble in the spectral form. What is the Shannon entropy over the spectral form of the states? (7 pts.)

PROBLEM 2: This problem has 2 parts:

Consider a qubit channel mapping $\mathcal{E}(\rho) = \frac{1}{2}\left(aI\,\mathrm{tr}(\rho) + bX\rho X + cY\rho Y + dZ\rho Z\right)$ with non-negative real constants a, b, c, d.

(1) What are the conditions on the constants a, b, c, d for the map to be (a) linear and (b) trace-preserving? (3 pts.)

(2) Suppose you are given a input density matrix ρ with Bloch vector (r_x, r_y, r_z). We wish to smoothly shrink ρ using $\mathcal{E}$ to obtain the Bloch vector $(\mu_x r_x, \mu_y r_y, \mu_z r_z)$. Determine the parameters of $\mathcal{E}(\rho)$ to accomplish this mapping. (7 pts.)

PROBLEM 3: Suppose that there is a set of density operators ρ_k^S and a POVM $\{\Lambda_k^S\}$ that identifies the states with high probability i.e., $\forall k$ and $\epsilon > 0$, $\mathrm{tr}\left(\Lambda_k^S \rho_k^S\right) \geq 1 - \epsilon$. Construct a coherent measurement $U^{S \to SS'}$ having high probability of success i.e., $\langle \phi_k|^{RS} \langle k|^{S'} U^{S \to SS'} |\phi_k\rangle^{RS} \geq 1 - \epsilon$, where $|\phi_k\rangle^{RS}$ is a purification of ρ_k. (8 pts.)

PROBLEM 4: This problem has two parts:

(1) From first principles, derive the classical capacity of the binary erasure channel. (3 pts.)

(2) Derive the Kraus operators for a quantum binary erasure channel $\mathcal{E}(\rho) = (1-\epsilon)\rho + \epsilon\,|e\rangle\langle e|$, where $|e\rangle$ is not in the input Hilbert space. Obtain the isometric extension of this channel in Dirac form as well as in the matrix representation. (7 pts.)

PROBLEM 5:

Consider two independent random variables $X^{(n)}$ and $Y^{(n)}$ with marginal densities $p_{X^{(n)}}\left(x^{(n)}\right)$ and $p_{Y^{(n)}}\left(y^{(n)}\right)$. Prove that $P\left(\left(x^{(n)}, y^{(n)}\right) \in T_{\delta}^{X^{(n)}Y^{(n)}}\right) \leq 2^{-(nI(X;Y)-3c\delta)}$, where $T_{\delta}^{X^{(n)}Y^{(n)}}$ is the strong jointly typical set. (10 pts.)