Indian Institute of Science

Quantum Information Theory

Instructor: Shayan Srinivasa Garani
Homework #3, Fall 2023

Late submission policy: Points scored = Correct points scored $\times e^{-d}$, d = # days late

Assigned date: Oct. 18^{th}, 2023 **Due date:** Nov. 2^{nd}, 2023, in the class

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PROBLEM 1: (a) In the class, we discussed two different norm definitions over the operator M i.e., (a) $||M||_a = \sqrt{\text{tr}(|M|^2)}$ and (b) $||M||_b = \max\limits_{U:\text{unitary}} |\text{tr}(UM)|$. Are these norms the same? If not, what is their relationship? Justify all the details mathematically. (b) From first principles prove the Cauchy-Schwarz inequality for any two operators A and B over the Hilbert-Schmidt norm i.e., $\text{tr}(A^\dagger B) \leq \sqrt{\text{tr}(A^\dagger A)\text{tr}(B^\dagger B)}$. (8 pts.)

PROBLEM 2: (a) Quantum gates can be realized using unitary transformations. Suppose that operator $\tilde{U}$ approximates U within an error δ in the Hilbert-Schmidt norm sense. Consider a unitary operator $\tilde{V} = \tilde{V}_{n-1}...\tilde{V}_0$ for approximating $V = V_{n-1}...V_0$, where $||\tilde{V}_j - V_j|| \leq \delta_j$. What can you say about $||\tilde{V} - V||$? (b) Solve exercise problem 9.22 from Nielsen and Chuang. (10 pts.)

PROBLEM 3: Consider a generalization of the binary hypothesis problem to the multi-class hypothesis detection case. Suppose we have an ensemble $\{p_X(x), \rho_x\}$ that needs to be detected with mimimum probability of misclassification using POVMs $\{\Lambda_x\}$. Derive an expression for the mimimum probability of error by formulating the problem within an optimization framework. Specify all the constraints on the operators within your setup carefully. (5 pts.)

PROBLEM 4: Consider a single qudit $|\psi\rangle$ going through two instances of noisy channels, such as (a) a phase damping channel, and (b) a depolarizing channel. What is the minimum fidelity achieved over these channels? Over what quantum states are the minimum fidelities achieved? (5 pts.)

PROBLEM 5: Suppose $\mathcal{E}$ is a *strictly* contractive trace-preserving quantum map over density operators ρ and σ i.e., $||\mathcal{E}(\sigma) - \mathcal{E}(\rho)|| < ||\rho - \sigma||$. Prove that $\mathcal{E}$ has a fixed point. (7 pts.)