Quantum Information Theory
Assignment-3 solutions

by Samarth Kashyap

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9.1.8

$$\|\mathcal{N}(\rho)-\mathcal{N}(\sigma)\| \leq \|\rho-\sigma\|$$

writing $\mathcal{N}(\rho) = \sum_j M_j \rho M_j^\dagger$

we have an isometric extension to the environment

$$\mathcal{U}_{\mathcal{N}}^{A\to BE} = \sum_j M_j \otimes |j\rangle^E$$

We know that

$$\|\mathcal{N}(\rho)-\mathcal{N}(\sigma)\| \leq \|\mathcal{U}_{\mathcal{N}}^{A\to BE}(\rho)-\mathcal{U}_{\mathcal{N}}^{A\to BE}(\sigma)\|$$

Since $\mathcal{U}_{\mathcal{N}}^{A\to BE}$ is isometric,

$$\left\| \mathcal{U}_{\mathcal{N}}^{A\to BE}(\rho) - \mathcal{U}_{\mathcal{N}}^{A\to BE}(\sigma) \right\| = \| \rho - \sigma \|$$

$$\Rightarrow \| \mathcal{N}(\rho) - \mathcal{N}(\sigma) \| \leq \| \rho - \sigma \|$$

9.2.4

To show $F(\rho, \sigma) = \mathrm{Tr}\left\{ \sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}} \right\}^2$

$$F(\rho,\sigma) = \| \sqrt{\rho}\sqrt{\sigma} \|^2$$

$$= \mathrm{Tr}\left\{ \sqrt{\sqrt{\rho}\sqrt{\sigma}\left(\sqrt{\rho}\sqrt{\sigma}\right)^\dagger} \right\}^2$$

$$= \mathrm{Tr}\left\{ \sqrt{\sqrt{\rho}\sqrt{\sigma}\sqrt{\sigma}^\dagger\sqrt{\rho}^\dagger} \right\}^2$$

Since $\sqrt{\rho}$ & $\sqrt{\sigma}$ are Hermitian,

$$F(\rho,\sigma) = \mathrm{Tr}\left(\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}} \right)^2$$

9.2.5

$$F(\rho,\sigma) = \max_{\mathcal{U}} \left| \langle \phi_\rho | \mathcal{U}^R \otimes \mathbb{1}^A | \phi_\sigma \rangle^{RA} \right|^2$$

$$= \max_{\mathcal{U}} \left| \langle \phi_\rho | \mathcal{U}^R \otimes \mathcal{U}^\dagger \mathcal{U} | \phi_\sigma \rangle^{RA} \right|^2$$

$$= \max_{\mathcal{U}} \left| \langle \phi_\rho | (\mathbb{1} \otimes \mathcal{U}^\dagger)(\mathcal{U}^R \otimes \mathbb{1}^A)(\mathbb{1} \otimes \mathcal{U}) | \phi_\sigma \rangle^{RA} \right|^2$$

$$= \max_{\mathcal{U}} \left| \langle \phi_{\mathcal{U}\rho\mathcal{U}^\dagger} | \mathcal{U}^R \otimes \mathbb{1}^A | \phi_{\mathcal{U}\sigma\mathcal{U}^\dagger} \rangle^{RA} \right|^2$$

$$= F(\mathcal{U}\rho\mathcal{U}^\dagger, \mathcal{U}\sigma\mathcal{U}^\dagger)$$

9.2.6

We know $F(\rho^{AB}, \sigma^{AB}) \leq F(\rho^A, \sigma^A)$

Consider isometric extension $\mathcal{U}_N^{A\to BE}$

$$F(\mathcal{U}_N^{A\to BE}(\rho), \mathcal{U}_N^{A\to BE}(\sigma)) \leq F(N(\rho), N(\sigma))$$

But $\mathcal{U}_N^{A\to BE}(\rho) = \mathcal{U}_N^{A\to BE} \rho \, \mathcal{U}_N^{A\to BE\,T}$

$$\mathcal{U}_{\mathcal{N}}^{A\to BE}(\sigma) = \mathcal{U}_{\mathcal{N}}^{A\to BE}\, \sigma\, \mathcal{U}_{\mathcal{N}}^{A\to BE^\dagger}$$

From 9.2.5, $F(\rho,\sigma) = F(\mathcal{U}\rho\mathcal{U}^\dagger, \mathcal{U}\sigma\mathcal{U}^\dagger)$

$$\therefore F(\rho,\sigma) \leq F(\mathcal{N}(\rho), \mathcal{N}(\sigma))$$

9.4.2

$$\mathrm{Tr}\{\Lambda_k^A \rho_k^A\} \geq 1-\epsilon \qquad \forall k$$

$$\mathrm{Tr}_R\{\rho_k^{RA}\} = \rho_k^A$$

$$\rho_k^{RA} = |\phi_k\rangle\langle\phi_k|^{RA}$$

Let

$$\mathcal{D}^{A\to AK}(\rho_k^{RA}) = (\mathbb{1}^R \otimes \mathcal{U}^{AK})(\rho_k^{RA} \otimes |k\rangle\langle k|)(\mathbb{1}^R \otimes \mathcal{U}^{AK^\dagger})$$

$$\mathcal{U}_{AK} = \sum_k \sqrt{\Lambda_k^A} \otimes |k\rangle\langle k|^K$$

$$\mathcal{D}^{A \to AK}(\phi_k^{RA}) = \sqrt{\Lambda_k^A}\, \rho_k^{RA} \sqrt{\Lambda_k^A} \otimes |k\rangle\langle k|^K$$

$$\left\| \mathcal{D}^{A\to AK}(\phi_k^{RA}) - \rho_k^{RA} \otimes |k\rangle\langle k|^K \right\|$$

$$= \left\| \sqrt{\Lambda_k^A}\, \rho_k^{RA} \sqrt{\Lambda_k^A} \otimes |k\rangle\langle k|^K - \rho_k^{RA} \otimes |k\rangle\langle k|^K \right\|$$

$$= \left\| \sqrt{\Lambda_k^A}\, \rho_k^{RA} \sqrt{\Lambda_k^A} - \rho_k^{RA} \right\| \qquad -①$$

By gentle operator lemma,

If $\mathrm{Tr}\{\Lambda_k^A \rho_k^A\} \geq 1-\epsilon \quad \forall k$,

$$\left\| \rho_k^A - \sqrt{\Lambda_k^A}\, \rho_k^A \sqrt{\Lambda_k^A} \right\| \leq 2\epsilon \qquad -②$$

Therefore, from ① & ②,

$$\left\| \mathcal{D}^{A\to AK}(\phi_k^{RA}) - \rho_k^{RA} \otimes |k\rangle\langle k|^K \right\| \leq 2\epsilon \quad \forall k$$

11.6.3

$$I(A;B) = H(A) - H(A|B)$$
$$\leq H(A) + |H(A|B)|$$

Since $H(A|B) \leq H(A) \leq \log d_A$,

$$I(A;B) \leq 2 \log d_A$$

lly from $I(A;B) = H(B) - H(B|A)$

$$I(A;B) \leq 2 \log d_B$$

$$\therefore I(A;B) \leq 2 \min\{\log d_A, \log d_B\}$$

11.6.4

$$I(R;B)_\phi = H(R) + H(B) - H(RB)$$

$$I(R;E)_\phi = H(R) + H(E) - H(RE)$$

Since $|\psi\rangle^{RA}$ & $|\phi\rangle^{RBE}$ are pure,

$$H(R)_\psi = H(A)_\psi \qquad H(RA)_\psi = 0$$

$$H(RB)_\phi = H(E) \qquad H(RE)_\phi = H(B)_\phi$$

$$I(R;B)_\phi + I(R;E)_\phi = 2H(R)_\phi + H(B)_\phi + H(E)_\phi - H(RB)_\phi - H(RE)_\phi$$

$$= 2H(R)_\phi$$

$$I(R;A)_\psi = H(R)_\psi + H(A)_\psi - H(RA)_\psi$$

$$= 2H(R)_\phi$$

Since entropy is invariant under isometry, these are equal.

11.6.5

Since $|\psi\rangle^{SRA}$ & $|\phi\rangle^{SRBE}$ are pure,

$$H(RA)_\psi = H(S)_\psi \qquad H(RS)_\psi = H(A)_\psi$$

$$H(RB)_\phi = H(SE)_\phi \qquad H(RSE)_\phi = H(B)_\phi$$

Therefore,

$$I(R;A)_\psi = H(R)_\psi + H(A)_\psi - H(S)_\psi$$

$$I(R;S)_\psi = H(R)_\psi + H(S)_\psi - H(A)_\psi$$

$$I(R;B)_\phi = H(R)_\phi + H(B)_\phi - H(SE)_\phi$$

$$I(R;SE)_\phi = H(R)_\phi + H(SE)_\phi - H(B)_\phi$$

$$I(R;A)_\psi + I(R;S)_\psi = 2H(R)_\psi$$

$$I(R;B)_\phi + I(R;SE)_\phi = 2H(R)_\psi$$

Again entropy invariant under isometry.

11.7.6

$$I(A;C|B) \geq 0$$

$$H(A|B) + H(C|B) - H(AC|B) \geq 0$$

$$H(AB) + H(BC) - H(ABC) - H(B) \geq 0$$

$$\Rightarrow H(AB) + H(BC) \geq H(ABC) - H(B)$$

11.8.5

$$D(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2)$$

$$= \mathrm{Tr}\{(\rho_1 \otimes \rho_2)(\log(\rho_1 \otimes \rho_2) - \log(\sigma_1 \otimes \sigma_2))\}$$

$$\log A \otimes B = \log (A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes B)$$

$$= \log A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \log B$$

$$D(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = \mathrm{Tr}\left\{(\rho_1 \otimes \rho_2)\left((\log \rho_1 - \log \sigma_1) \otimes \mathbb{1} + \mathbb{1} \otimes (\log \rho_2 - \log \sigma_2)\right)\right\}$$

$$= \mathrm{Tr}\{\rho_1(\log \rho_1 - \log \sigma_1)\} + \mathrm{Tr}\{\rho_2(\log \rho_2 - \log \sigma_2)\}$$

$$= D(\rho_1 \| \sigma_1) + D(\rho_2 \| \sigma_2)$$

2. Alice prepares $\sigma^{AB} = \sum_x p_x |x\rangle\langle x|^A \otimes \rho_x^B$

We have $\rho^B = \sum_x p_x \rho_x^B$, $H(B|A) = \sum_x p_x H(\rho_x^B)$

$$I(A;B)_\sigma = H(B)_\sigma - H(B|A)_\sigma$$

$$= H\left(\sum_x p_x \rho_x^B\right) - \sum_x p_x H(\rho_x^B) = \chi$$

Bob only has access to B and any measurements he makes are a map $\mathcal{N}^{B \to X}$ on ρ_B. By the data-processing inequality,

$$I(A;B)_\sigma \geq I(A;X)_\tau$$

where $\tau^{AX} = \mathcal{N}^{B\to X}(\sigma^{AB})$

$$\Rightarrow I(A;X)_\tau \leq \mathcal{X} \qquad \forall\ \mathcal{N}^{B\to X}$$

$$\therefore I_{acc} = \max_{\{\Lambda_m^x\}} I(A;X) \leq \max_{\mathcal{N}^{B\to X}} I(A;X) \leq \mathcal{X}$$

3. $|X_1\rangle = |0\rangle \qquad |X_2\rangle = \frac{1}{\sqrt{3}}\left[|0\rangle + \sqrt{2}|1\rangle\right]$

$$|X_3\rangle = \frac{1}{\sqrt{3}}\left[|0\rangle + \sqrt{2}e^{2\pi i/3}|1\rangle\right]$$

$$|X_4\rangle = \frac{1}{\sqrt{3}}\left[|0\rangle + \sqrt{2}e^{4\pi i/3}|1\rangle\right]$$

$$\rho = \frac{1}{4}\sum_i |X_i\rangle\langle X_i|$$

$$= \frac{1}{4}|0\rangle\langle 0| + \frac{1}{12}\left[3|0\rangle\langle 0| + 6|1\rangle\langle 1| + \sqrt{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)\left(1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}}\right)\right]$$

$$1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$$

$$\therefore \rho = \frac{\mathbb{1}}{2} = \pi$$

$$I_{acc}(\rho) \leq \chi(\rho)$$

$$= H(\rho) - \sum_i p_i H(\rho_i)$$

Since $\rho_i = |X_i\rangle\langle X_i|$ is a pure state, $H(\rho_i) = 0$

$$\therefore I_{acc}(\rho) \leq H(\rho) = \mathrm{Tr}\left(\sqrt{\frac{\mathbb{1}}{4}}\right) = \mathrm{Tr}\left(\frac{\mathbb{1}}{2}\right)$$

$$I_{acc}(\rho) \leq 1 \text{ bit}$$

Now, consider a POVM $\{\Lambda_m = |Y_m\rangle\langle Y_m|\}$.

The mutual information is

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

$$= \sum_i p(X_i) \log \frac{1}{p(X_i)} + \sum_j p(Y_j) \log \frac{1}{p(Y_j)}$$

$$- \sum_{i,j} p(X_i, Y_j) \log \frac{1}{P(X_i, Y_j)}$$

$$p(X_i) = \sum_j p(X_i, Y_j) \qquad p(Y_j) = \sum_i p(X_i, Y_j)$$

$$\therefore I(X;Y) = \sum_{i,j} p(X_i, Y_j) \log \frac{p(X_i, Y_j)}{p(X_i)\, p(Y_j)}$$

We know $p(X_i) = 1/4 \ \forall i$.

$$p(X_i, Y_j) = p(Y_j | X_i)\, p(X_i)$$

$$= \frac{1}{4} \mathrm{Tr}\{\Lambda_j \rho_i^x\}$$

$$= \frac{\langle y_j | x_i \rangle}{4} \mathrm{Tr}\left(|y_j\rangle\langle x_i| \right)$$

For statevectors $|\phi\rangle$ & $|\psi\rangle$, $\mathrm{Tr}\left(|\phi\rangle\langle\psi|\right) = \langle\psi|\phi\rangle$

$$\Rightarrow p(x_i, y_j) = \frac{|\langle y_j | x_i \rangle|^2}{4}$$

$$p(y_j) = \sum_k p(x_k, y_j) = \sum_k \frac{|\langle y_j | x_k \rangle|^2}{4}$$

$$I(X;Y) = \frac{1}{4} \sum_{ij} |\langle y_j | x_i \rangle|^2 \log \frac{4 |\langle y_j | x_i \rangle|^2}{\sum_k |\langle y_j | x_k \rangle|^2}$$

Consider $|y_j\rangle = ZX \frac{|x_j\rangle}{\sqrt{2}}$ (orthogonal to $|x_i\rangle$)

$|y_1\rangle = \frac{1}{\sqrt{2}} |1\rangle$ $\quad |y_2\rangle = \sqrt{\frac{1}{6}} \left[\sqrt{2}\,|0\rangle - |1\rangle \right]$

$|y_3\rangle = \sqrt{\frac{1}{6}} \left[\sqrt{2}\,|0\rangle - e^{2\pi i/3} |1\rangle \right]$ $\quad |y_4\rangle = \sqrt{\frac{1}{6}} \left[\sqrt{2}\,|0\rangle - e^{4\pi i/3} |1\rangle \right]$

$\langle y_j | x_i \rangle$ is given by:

$i \backslash j$	1	2	3	4
1	0	$1/\sqrt{3}$	$\frac{e^{4\pi i/3}}{\sqrt{3}}$	$\frac{e^{2\pi i/3}}{\sqrt{3}}$
2	$1/\sqrt{3}$	0	$\frac{1-e^{4\pi i/3}}{3}$	$\frac{1-e^{2\pi i/3}}{3}$
3	$\frac{e^{2\pi i/3}}{\sqrt{3}}$	$\frac{1-e^{2\pi i/3}}{3}$	0	0
4	$\frac{e^{4\pi i/3}}{\sqrt{3}}$	$\frac{1-e^{4\pi i/3}}{3}$	0	0

and $|\langle Y_j | X_i \rangle|^2$ is given by

$i \backslash j$	1	2	3	4
1	0	$1/3$	$1/3$	$1/3$
2	$1/3$	0	$1/3$	$1/3$
3	$1/3$	$1/3$	0	$1/3$
4	$1/3$	$1/3$	$1/3$	0

From these,

$$I(X;Y) \approx 0.415 \text{ bits}$$