

(1) (a) The probability of correct decision is

$$\begin{aligned}
 P_c &= \sum_{i=1}^k \frac{1}{k} \text{tr} (M_i^+ M_i S_i) \\
 &\leq \sum_{i=1}^k \frac{1}{k} \text{tr} (M_i^+ M_i) \\
 &= \frac{1}{k} \text{tr} \left(\sum_{i=1}^k M_i^+ M_i \right) \\
 &= \frac{1}{k} \text{tr} (I) = \frac{d}{k}
 \end{aligned}$$

(b) Note from the 1st equality of (a)

$$\begin{aligned}
 P_c &= \frac{1}{k} \sum_{i=1}^k \text{tr} (M_i^+ M_i S_i) \\
 &\leq \frac{1}{k} \sum_{i=1}^k \|M_i^+ M_i\| \|S_i\|
 \end{aligned}$$

Choose $i^* = \arg \max_i$

$$\begin{aligned}
 &\leq \frac{1}{k} \max_i \|S_i\| \underbrace{\sum_{i=1}^k \|M_i^+ M_i\|}_d \\
 &= \frac{d}{k} \max_i \|S_i\|
 \end{aligned}$$

The strong subadditivity implies

$$H(C) + H(ABC) \leq H(AC) + H(BC)$$

Recall in terms of $E \cup F$, $E \cap F$, it is helpful.

$$H(ABC) - H(C) \leq H(AC) - H(C) + H(BC) - H(C)$$

(Subtracting $2H(C)$ from both sides)

$$H(AB|C) \leq H(A|C) + H(B|C)$$

(Conditional entropy defn)

□

From the concavity of entropy

$$H(\gamma^A) \geq (1-\epsilon) H(A)_P + \epsilon H(A)_{\tilde{P}}$$

Having a classical variable X can only increase entropy.

$$H(A)_\gamma \leq H(AX)_\gamma$$

$$\leq H(A|X)_\gamma + H(X)$$

(From chain rule)

$$H(X) = H_2(\epsilon)$$

$$H(A|X)_\gamma = (1-\epsilon) H(A)_P + \epsilon H(A)_{\tilde{P}}$$

$$\leq \epsilon \left(H(A)_{\tilde{P}} + H(A)_P \right)$$

$$\leq 2\epsilon \log(d_A) \quad \left(\begin{array}{l} \text{III} \\ \text{dar} \\ \text{proof} \\ \text{done in} \\ \text{class} \end{array} \right)$$

$$\leq 2\epsilon \log(d_A) + 2H(\epsilon)$$

Using (2) & (1), the result follows. (2)



Let us work out Part (A) similar to what we did in the class. This is Fannes in eq.

We know $\| \rho^A - \sigma^A \| \leq \epsilon$.

Suppose $\| \rho^A - \sigma^A \| = \epsilon$ (worst case)

Define the following density operators

$$\tilde{\rho}^A \triangleq \frac{1}{\epsilon} | \rho^A - \sigma^A |$$

$$\tilde{\sigma}^A \triangleq \frac{1-\epsilon}{\epsilon} (\rho^A - \sigma^A) + \tilde{\rho}^A$$

So $\tilde{\rho}^A$ is a valid density operator.

Construct a classical quantum state

$$\gamma^{XA} \triangleq (1-\epsilon) |0\rangle\langle 0|^X \otimes \rho^A + \epsilon |1\rangle\langle 1|^X \otimes \tilde{\rho}^A$$

Note that $\text{tr}_X (\gamma^{XA}) = (1-\epsilon) \rho^A + \epsilon \tilde{\rho}^A$

$\gamma^A \uparrow$ \uparrow Complex combination

(2) Consider any two quantum states ρ^{AB} and σ^{AB} , $\|\rho^{AB} - \sigma^{AB}\|_1 \leq \epsilon$.

From the monotonicity result under discarding of subsystems

$$\|\rho^A - \sigma^A\|_1 \leq \|\rho^{AB} - \sigma^{AB}\|_1 \leq \epsilon$$

(From problem statement)

Consider $|\mathcal{I}(A; B)_\rho - \mathcal{I}(A; B)_\sigma|$

$$= |H(A)_\rho - H(A|B)_\rho - (H(A)_\sigma - H(A|B)_\sigma)|$$

(Defn of mutual info.)

$$\leq |H(A)_\rho - H(A)_\sigma| + |H(A|B)_\rho - H(A|B)_\sigma|$$

Part (A)

Triangle inequality
Part (B)

From Alicki: Fannes inequality done in class

$$|H(A|B)_\rho - H(A|B)_\sigma| \leq 4\epsilon \log(d_A) + 2H_2(\epsilon)$$

① This sorts out Part (B).

(2) (b) We are given an ensemble

$$\{ P_x(x), |\psi_x\rangle \}$$

$$\rho \doteq \sum_x P_x(x) |\psi_x\rangle \langle \psi_x|$$

Let's start with a classical common randomness state

$$\sum_x P_x(x) |x\rangle \langle x| \otimes |x\rangle \langle x|_{\text{System B}}$$

Apply a preparation map over (1) to get

$$\sum_x P_x(x) |x\rangle \langle x| \otimes |\psi_x\rangle \langle \psi_x|_{\text{System B}}$$

From quantum data processing theorem

$$I(X; B) \geq I(X; B_1)$$

$$H(X) - H(X|B) \geq H(B_1) - H(B_1|X)$$

$$0 \geq H(S) - H(X)$$

Shared randomness

$$H(X) \geq H(S)$$

For orthogonal to be eigen values of ρ happen $\Rightarrow H(S) = H(X)$



3) a) In trying to bring in coherent information, you must clearly bring in your understanding that "quantum correlations" are what the channel is trying to maximize.

Set up:

Consider a state $\phi_{AA_1A_2'}$, input to a tensor product channel;

$$U_{N_1} \begin{matrix} A_1' \rightarrow B_1 E_1 \\ A_2' \rightarrow B_2 E_2 \end{matrix} (\phi_{AA_1A_2'}) = \sigma_{AB_1E_1A_2'}$$

$$U_{N_2} (\phi_{AA_1A_2'}) = \theta_{AB_2E_2A_1'}$$

$$U_{N_1} \approx N_1 \otimes I_{A_2'} \quad \left. \begin{matrix} A_1 \rightarrow B_1 E_1 \\ A_2' \rightarrow B_2 E_2 \end{matrix} \right\} \text{equivalent actions}$$

$$U_{N_2} = I_{A_1'} \otimes N_2 \quad (\text{pure})$$

$$\psi = \psi_{Q(N_1)} + Q(N_2)$$

$$= I(A > B_1 A_2') + I(A > B_2 A_1')$$

$$= H(B_1)_\sigma - H(AA_2'B_1)_\sigma + H(B_2)_\theta - H(AA_1'B_2)_\theta$$

$$= H(B_1 B_2)_\psi - H(E_1)_\sigma - H(E_2)_\theta$$

$$= H(B_1 B_2)_\psi - H(AB_1 B_2)_\rho \quad (\because \text{product states})$$

$$\leq I(A > B_1 B_2)_\psi \quad (\text{over choice of general form of } \psi)$$

(b) The Helms information is
Concave in i/p distribution with
fixed signaling states. — (1)

It is convex in i/p distribution
with fixed distribution (2)

Cases (1) or (2) are fine as long
as one quantity is fixed.

If not, one must work out an
alternating optimization program
to solve the global optimization
problem, which can be quite a
task in higher dimensions

