

# Quantum Information Theory

## Homework-2 solutions

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1.

4.1.8

$$\text{We have } \rho_j = \frac{1}{2} (\mathbb{1} + \vec{r}_j \cdot \vec{\sigma})$$

$$\rho = \sum_j p(j) \rho_j$$

$$= \frac{1}{2} \sum_j p(j) (\mathbb{1} + \vec{r}_j \cdot \vec{\sigma})$$

$$\text{Since } \sum_j p(j) = 1,$$

$$\rho = \frac{1}{2} (\mathbb{1} + \sum_j p(j) \vec{r}_j \cdot \vec{\sigma})$$

$$\rho = \frac{1}{2} (\mathbb{1} + \vec{r} \cdot \vec{\sigma})$$

$$\therefore \vec{r} = \sum_j p(j) \vec{r}_j$$

4.9.2.

$$\rho = \sum_x p(x) \rho_x$$

$$P_{\text{success}} = \sum_x p(x) \text{Tr}(\rho_x \Lambda_x)$$

$$T \geq p(x) \rho_x \quad \forall x.$$

$$P_{\text{success}} = \sum_x \text{Tr}(p(x) \rho_x \Lambda_x)$$

$$P_{\text{success}} \leq \sum_x \text{Tr}(T \Lambda_x)$$

$$\leq \text{Tr}(T \sum_x \Lambda_x)$$

$$P_{\text{success}} \leq \text{Tr}(T)$$

Now, for encoding  $n$  bits in a  $d$ -dimensional subspace,

$$P(x)\rho_x = \begin{cases} 2^{-n} \rho_x, & x \in \{0,1\}^n \\ 0 & \text{otherwise} \end{cases}$$

Since  $\rho_x$  has eigenvalues  $\leq 1$

$$2^{-n} \mathbb{1} \geq 2^{-n} \rho_x \quad \forall x.$$

$$\Rightarrow P_{\text{success}} \leq \text{Tr}(2^{-n} \mathbb{1})$$

$$P_{\text{success}} \leq d 2^{-n}$$

## 4.4.7

Consider

$$\rho' = \frac{1}{4}\rho + \frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z$$

$$\text{Writing } \rho = \frac{1}{2} \left( \mathbb{1} + \sum_i r_i \sigma_i \right),$$

$$\sigma_j \rho \sigma_j = \frac{\sigma_j}{2} \left( \sigma_j + r_j \mathbb{1} + \sum_{i \neq j} r_i \sigma_i \sigma_j \right)$$

$$= \frac{1}{2} \left( \mathbb{1} + r_j \sigma_j - \sum_{i \neq j} r_i \sigma_i \right)$$

$$\Rightarrow \rho' = \frac{1}{4}\rho + \frac{1}{4} \left( \frac{3\mathbb{1}}{2} - \frac{r_x X + r_y Y + r_z Z}{2} \right)$$

$$= \frac{\mathbb{1}}{2} = \pi$$



Now, the channel

$$\rho \rightarrow (1-p)\rho + p\bar{\rho}$$

can be written

$$\rho \rightarrow (1-p)\rho + \frac{p}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$$

$$\rho \rightarrow \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z)$$

4.4.10

$$\rho = \begin{bmatrix} 1-p & \eta \\ \eta^* & p \end{bmatrix}$$

$$= (1-p)|0\rangle\langle 0| + \eta|0\rangle\langle 1| + \eta^*|1\rangle\langle 0| + p|1\rangle\langle 1|$$

Kraus operators for ADC:

$$A_0 = \sqrt{\gamma} |0\rangle\langle 1| \quad A_1 = |0\rangle\langle 0| + \sqrt{1-\gamma} |1\rangle\langle 1|$$

$$\begin{aligned} \rho_0 &= A_0 \rho A_0^\dagger = \gamma |0\rangle\langle 1| \rho |1\rangle\langle 0| \\ &= \gamma \rho |0\rangle\langle 0| \end{aligned}$$

$$\begin{aligned} A_1 \rho A_1^\dagger &= (|0\rangle\langle 0| + \sqrt{1-\gamma} |1\rangle\langle 1|) \rho (|0\rangle\langle 0| + \sqrt{1-\gamma} |1\rangle\langle 1|) \\ &= (1-\rho) |0\rangle\langle 0| + \sqrt{1-\gamma} \eta |0\rangle\langle 1| + \sqrt{1-\gamma} \eta^* |1\rangle\langle 0| \\ &\quad + (1-\gamma) \rho |1\rangle\langle 1| \end{aligned}$$

∴ The ADC gives

$$\rho' = \begin{bmatrix} 1 - (1-\gamma)\rho & \sqrt{1-\gamma} \eta \\ \sqrt{1-\gamma} \eta^* & (1-\gamma)\rho \end{bmatrix}$$

## 4.4.11

Using the result from 4.4.10 for a general  $\rho$ ,

$$\rho = \begin{bmatrix} 1-p & \eta \\ \eta^* & p \end{bmatrix}$$

$$N_1(\rho) = \begin{bmatrix} 1 - (1-\gamma_1)p & \sqrt{1-\gamma_1} \eta \\ \sqrt{1-\gamma_1} \eta^* & (1-\gamma_1)p \end{bmatrix}$$

$$N_2(N_1(\rho)) = \begin{bmatrix} 1 - (1-\gamma_2)(1-\gamma_1)p & \sqrt{1-\gamma_2} \sqrt{1-\gamma_1} \eta \\ \sqrt{1-\gamma_2} \sqrt{1-\gamma_1} \eta^* & (1-\gamma_2)(1-\gamma_1)p \end{bmatrix}$$

$$\therefore N_2(\gamma_2) \circ N_1(\gamma_1) = N(\gamma_1 + \gamma_2 - \gamma_1 \gamma_2)$$

## 4.4.13

$$\rho \rightarrow \langle k | \rho | k \rangle |k\rangle \langle k| \otimes \sigma_k$$

Consider the state

$$|\psi\rangle^{AB} = \sum_{ij} a_{ij} |ij\rangle$$

We have

$$\rho_{AB} = \sum_{ijkl} a_{ij} a_{kl} |ij\rangle \langle kl|$$

where  $i, j, k, l \in \{0, 1\}$

$$\rho_B = \text{Tr}_A(\rho_{AB})$$

$$= \sum_{jk} a_{ij} a_{ik} |j\rangle \langle k|$$

$$\mathcal{N}(\rho_B) = \sum_{ijkl} a_{ij} a_{ik} \langle l|j\rangle \langle k|l\rangle |l\rangle \langle l|$$

$$= \sum_j a_{ij}^2 |j\rangle \langle j|$$

which is a separable state of the form  $\sum_j c_j |j\rangle \langle j|$   
 with  $c_j = \sum_i a_{ij}^2$

2.

(a)  $\rho \xrightarrow{U} (1-\epsilon)\rho + \epsilon |e\rangle\langle e|$

$$U: \mathcal{H}_2 \rightarrow \mathcal{H}_3 \quad |e\rangle \perp |0\rangle, |1\rangle$$

Consider

$$M_0 = \sqrt{1-\epsilon} \left( |0\rangle_3 \langle 0|_2 + |1\rangle_3 \langle 1|_2 \right)$$

$$M_1 = \sqrt{\epsilon} |e\rangle_3 \langle 0|_2$$

$$M_2 = \sqrt{\epsilon} |e\rangle_3 \langle 1|_2$$

$$M_0^\dagger M_0 = (1-\epsilon) \mathbb{1}_2 \quad M_1^\dagger M_1 = \epsilon |0\rangle\langle 0| \quad M_2^\dagger M_2 = \epsilon |1\rangle\langle 1|$$

$$\sum M_i^\dagger M_i = \mathbb{1}$$

$$M_0 \rho M_0^\dagger = (1-\epsilon) \left( |0\rangle_3 \langle 0|_2 + |1\rangle_3 \langle 1|_2 \right) \rho \left( |0\rangle_2 \langle 0|_3 + |1\rangle_2 \langle 1|_3 \right)$$

$$= (1-\epsilon) \sum_{\substack{i,j \\ =0,1}} \rho_{ij} |i\rangle_3 \langle j|_3$$

$$M_1 \rho M_1^\dagger = \epsilon \rho_{00} |e\rangle_3 \langle e|_3$$

$$M_2 \rho M_2^\dagger = \epsilon \rho_{11} |e\rangle_3 \langle e|_3$$

$$\sum_i M_i \rho M_i^\dagger = (1-\epsilon) \rho + \epsilon |e\rangle \langle e|$$

In  $\mathcal{H}_3 \otimes \mathcal{H}_3$ ,  $\rho$  would be  $\begin{pmatrix} \rho_{00} & \rho_{01} & 0 \\ \rho_{10} & \rho_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$  but we can ignore.

(b) generalized dephasing channel

A dephasing channel only modifies the phases while keeping diagonal elements same:

$$\rho \rightarrow (1-p) \rho + p Z \rho Z$$

In a  $d$ -dimensional Hilbert space there are  $d$  phases

$$Z(z)|j\rangle = \exp(i2\pi zj/d)|j\rangle \quad z \in \{0, \dots, d-1\}$$

∴ The  $d$ -dimensional depolarizing channel is

$$\rho \rightarrow \sum_{i=0}^{d-1} p_i Z(i)\rho Z(i)^\dagger$$

The Kraus operators are  $\{Z(z)\}$   
 $z \in \{0, \dots, d-1\}$

3.

8.21

$$(1) H = \lambda (a^\dagger b + b^\dagger a)$$

$$U = e^{-iH\lambda t}$$

Let us denote the state  $|n\rangle_a |k\rangle_b$  as  $|n, k\rangle$ . The action of  $H$ , and therefore  $U$ , on this should therefore preserve  $n+k$ .

$\therefore E_k = \langle k_b | U | 0_b \rangle$  will be of the form

$$E_k = \sum_n \langle n-k, k | U | n, 0 \rangle |n-k\rangle \langle n|$$

$$= \sum_n \langle 0, 0 | \frac{a^{n-k}}{\sqrt{(n-k)!}} \frac{b^k}{\sqrt{k!}} U \frac{a^\dagger n}{\sqrt{n!}} | 0, 0 \rangle |n-k\rangle \langle n|$$

$$= \sum_n \sqrt{\binom{n}{k}} \frac{1}{n!} \langle 0, 0 | a^{n-k} b^k U a^\dagger n U | 0, 0 \rangle |n-k\rangle \langle n|$$



Since  $U$  preserves  $n+k$ ,  $U|0,0\rangle = |0,0\rangle$ .

$$E_k = \sum_n \sqrt{\binom{n}{k}} \frac{1}{n!} \langle 00 | a^{n-k} b^k U a^\dagger U^\dagger | 00 \rangle |n-k\rangle \langle n|$$

$$U A U^\dagger = U A U U^\dagger U A U^\dagger U = (U A U^\dagger)^n$$

$$E_k = \sum_n \sqrt{\binom{n}{k}} \frac{1}{n!} \langle 00 | a^{n-k} b^k (U a^\dagger U^\dagger)^n | 00 \rangle |n-k\rangle \langle n|$$

Now,

$$U a^\dagger U^\dagger = e^{-iH\Delta t} a^\dagger e^{iH\Delta t}$$

We have

$$e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{[X^n, Y]}{n!}$$

$$X = -iH\Delta t \quad Y = a^\dagger$$

$$[X, Y] = -i \chi \Delta t b^\dagger$$

$$[X, [X, Y]] = -\chi^2 \Delta t^2 a^\dagger$$

$$\therefore U a^\dagger U^\dagger = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\chi \Delta t)^{2n} a^\dagger$$

$$-i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\chi \Delta t)^{2n+1} b^\dagger$$

$$U a^\dagger U^\dagger = \cos(\chi \Delta t) a^\dagger - i \sin(\chi \Delta t) b^\dagger$$

$$U a^{\dagger n} U^\dagger = \left( a^\dagger \cos(\chi \Delta t) - b^\dagger i \sin(\chi \Delta t) \right)^n$$

$$= \sum_{r=0}^n (-i)^r \binom{n}{r} a^{\dagger n-r} b^{\dagger r} \cos^{n-r}(\chi \Delta t) \sin^r(\chi \Delta t)$$

Writing  $\gamma = 1 - \cos^2(\chi \Delta t)$ ,

$$U a^{\dagger n} U^\dagger = \sum_{r=0}^n (-i)^r \binom{n}{r} a^{\dagger n-r} b^{\dagger r} \sqrt{(1-\gamma)^{n-r} \gamma^r}$$

Since the only non-zero terms of  $\bar{E}_k$  will be those with equal applications of  $a$  &  $a^\dagger$ , and  $b$  &  $b^\dagger$ , only the  $r=k$  terms will appear, giving

$$\bar{E}_k = (-i) \sum_n \binom{n}{k} \frac{\sqrt{(1-\gamma)^{n-r} \gamma^r}}{n!} \langle 00 | a^{n-k} b^k a^{\dagger n-k} b^{\dagger k} | 00 \rangle |n-k\rangle \langle n|$$

$$a^k a^{\dagger k} |0\rangle = \sqrt{k!} a^k |k\rangle = k! |0\rangle$$

We can discard the global phase & write

$$E_k = \sum_n \sqrt{\binom{n}{k}} \sqrt{(1-\gamma)^{n-k} \gamma^k} |n-k\rangle \langle n|$$

$$(2) \bar{E}_k = \langle k_b | U | 0_b \rangle$$

$$E_k^\dagger \bar{E}_k = \langle 0_b | U^\dagger | k_b \rangle \langle k_b | U | 0_b \rangle$$

$$\sum_k E_k^\dagger \bar{E}_k = \langle 0_b | U^\dagger \left( \sum_k |k_b\rangle \langle k_b| \right) U | 0_b \rangle$$

$$= \langle 0_b | u^\dagger u | 0_b \rangle$$

$$\sum_k E_k^\dagger E_k = \mathbb{1}$$

$$\text{Tr} \left( \sum_k (E_k \rho E_k^\dagger) \right) = \sum_k \text{Tr} (E_k \rho E_k^\dagger)$$

$$= \sum_k \text{Tr} (E_k^\dagger E_k \rho)$$

$$= \text{Tr} \left[ \left( \sum_k E_k^\dagger E_k \right) \rho \right]$$

$$\text{Tr} \left( \sum_k (E_k \rho E_k^\dagger) \right) = \text{Tr}(\rho)$$

8.23

$$|\psi\rangle = a|01\rangle + b|10\rangle$$

$$\rho = |a|^2 |01\rangle\langle 01| + |b|^2 |10\rangle\langle 10|$$

$$+ ab^* |01\rangle\langle 10| + a^* b |10\rangle\langle 01|$$

$$E_0 = |0\rangle\langle 0| + \sqrt{1-\gamma} |1\rangle\langle 1|$$

$$E_1 = \sqrt{\gamma} |0\rangle\langle 1|$$

$$(E_{AD} \otimes \mathbb{1}) \rho = \sum_{i=0,1} (E_i \otimes \mathbb{1}) \rho (E_i \otimes \mathbb{1})$$

$$= |a|^2 |01\rangle\langle 01| + (1-\gamma) |b|^2 |10\rangle\langle 10|$$

$$+ ab^* \sqrt{1-\gamma} |01\rangle\langle 10| + a^* b \sqrt{1-\gamma} |10\rangle\langle 01|$$

$$+ \gamma |b|^2 |00\rangle\langle 00|$$

$$\begin{aligned}
 (\mathcal{E}_{AD} \otimes \mathcal{E}_{AD}) \rho &= \sum_{i=0,1} (\mathbb{1} \otimes E_i) (\mathcal{E}_{AD} \otimes \mathbb{1}) \rho (\mathbb{1} \otimes E_i^\dagger) \\
 &= (1-\gamma) (|a|^2 |01\rangle\langle 01| + |b|^2 |10\rangle\langle 10| \\
 &\quad + a^* b |110\rangle\langle 01| + a b^* |01\rangle\langle 10|) + \gamma |00\rangle\langle 00|
 \end{aligned}$$

$$(\mathcal{E}_{AD} \otimes \mathcal{E}_{AD}) \rho = (1-\gamma) \rho + \gamma |00\rangle\langle 00|$$

which is given by the Kraus operators

$$E_0^{dr} = \sqrt{1-\gamma} \mathbb{1}$$

$$E_1^{dr} = \sqrt{\gamma} [ |00\rangle\langle 01| + |00\rangle\langle 10| ]$$

