Introduction to wavelets

$e^{-j2\pi/N}$ τ

Motivation : The DFT provides uniform / equal frequency resolution & is an useful tool for analyzing the spectral content.

Qns.

a) Can we identify short bursts of high frequency signals over low-frequency signals ?

b) Can we approximate a signal by playing with the signal resolution "non uniformly" over the entire spectrum ?

Idea :

1) Use basis functions of 'different widths' to expand a signal across various scales (i.e., spaces)

2) In other words, project a signal onto a whole series of spaces with different resolution

Questions :

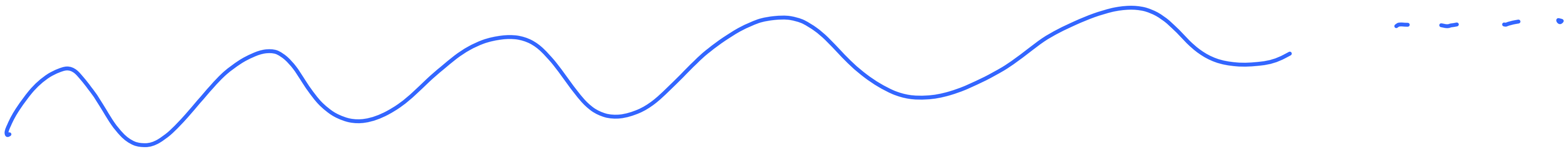
1) Can we reconstruct the signal perfectly ?

2) What are the properties for such a basis across scales ?

These ideas lead us to the concept of "wavelets"

Unlike $\sin(\cdot)$ & $\cos(\cdot)$ that have infinite support

$\sin(\omega t)$



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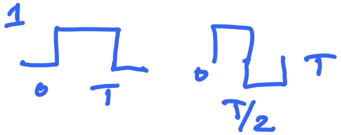
Pulses



(Finite support)

wavelets are pulses of short duration i.e., time localized and can provide different spectral information at different time locations of the signal

Multiresolution Property

Definition: Let V_j, $j = \ldots, -2, -1, 0, 1, 2, \ldots$ be a sequence of subspaces of functions in $L^2(\mathbb{R})$. The collection of spaces $\{V_j, j \in \mathbb{Z}\}$ is called a "multiresolution analysis" with a scaling function ϕ with the following properties.

1. **Nesting**: $V_j \subset V_{j+1}$

   i.e., $\ldots V_0 \subset V_1 \subset V_2 \ldots$

MEANING

(Every function in L^2 has a representation using elements in one of the nested sub spaces)

2. **Closure**: $\text{closure}\left(\bigcup_{j \in \mathbb{Z}} V_j\right) = L^2(\mathbb{R})$

   i.e., the closure of the set of spaces covers $L^2(\mathbb{R})$

3. <u>Shrinking</u> :

$$\bigcap_{j \in Z} v_j = \{0\}$$

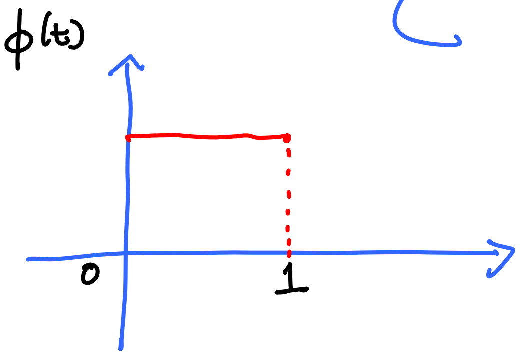
4. <u>Scaling</u> : If $f(t) \in V_j$, then $f(2^{-j} t) \in V_0$

5. <u>Shift orthonormality</u> :
The function $\phi(t) \in V_0$ and $\{\phi(t-k) ; k \in Z\}$ is an orthonormal basis for V_0.
i.e., $\langle \phi(t), \phi(t-n) \rangle = 0$

Let us see this through some examples.

Defn : The Haar scaling function is defined as

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{else} \end{cases}$$

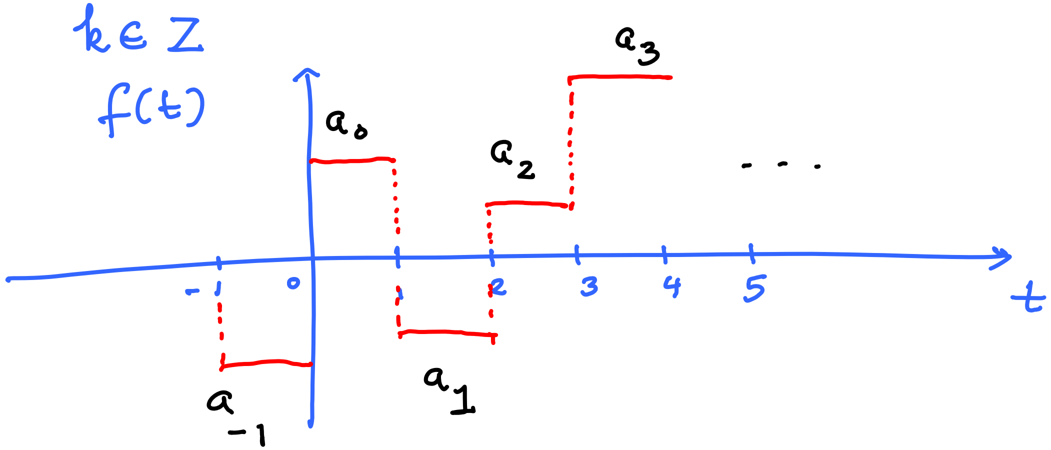


This function has finite/compact support

$\phi(t-k)$ is $\phi(t)$ translated by k units 'right' if k is a +ve integer

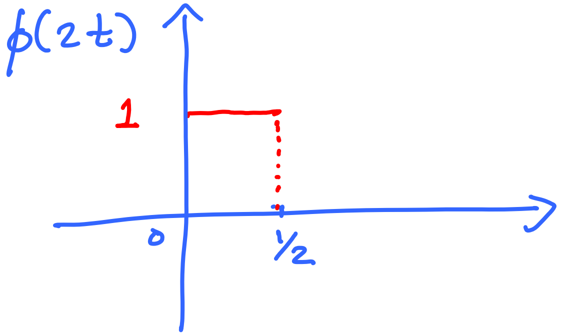
V_0 is a space of all functions of the form

$$f(t) = \sum_{k \in \mathbb{Z}} a_k \, \phi(t-k) \qquad a_k \in \mathbb{R}$$

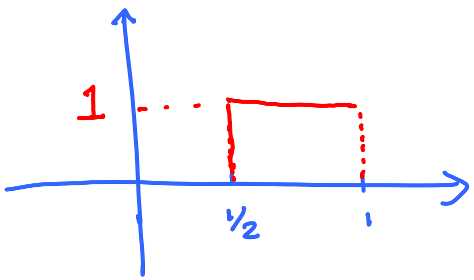


Let us go a little further ...

We shall start with a structure @ half the time duration
i.e., we start with $\phi(2t)$



Consider $\phi(2t - k) = \phi\left(2\left(t - k/2\right)\right)$



$\Leftarrow$ This is essentially translation of $\phi(2t)$ to the right $k = 1$ case

Geometrically, V_1 is the space of all functions of the form

$$\sum_{k \in \mathbb{Z}} a_k \, \phi(2t-k) \qquad a_k \in \mathbb{R}.$$

The possible discontinuities exist at half integer multiples

i.e., $0, \pm 1/2, \pm 1, \ldots$

Let us carefully analyze what the nesting property is:

$\phi(2t) \in V_1$ but $\notin V_0$ since $\phi(2t)$ is discontinuous @ $t = 1/2$

$V_0 \subset V_1$

Discontinuities in $V_0 = \{0, \pm 1, \pm 2, \ldots\}$

Discontinuities in $V_1 = \{0, \pm 1/2, \pm 1, \pm 3/2, \ldots\}$

$\Rightarrow$ Any function in V_0 is also contained in V_1 but not the other way.

Going forward,

V_j has all the information up to a resolution 2^{-j}

i.e., as $j \uparrow$, resolution is finer !

Ponder : Imagine functions of the form $f(a^j t)$
where a is a real no.

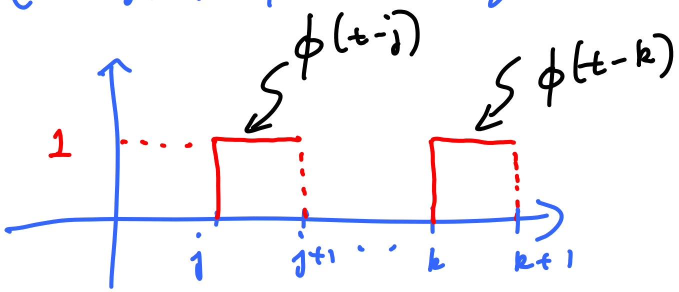
We need a procedure to do 'signal decomposition'

Let us see the intuition here before we proceed formally.

Starting with V_0,

$$\| \phi(t-k) \|_{L^2}^2 = \int_{-\infty}^{\infty} \phi^2(t-k)\, dt = \int_{k}^{k+1} 1^2\, dt = 1$$

$$\langle \phi(t-j), \phi(t-k) \rangle = \int_{-\infty}^{\infty} \phi(t-j)\, \phi(t-k)\, dt = 0 \quad j \neq k$$



Theorem: The set of functions $\{ 2^{j/2} \phi(2^j t - k), k \in Z \}$ form an orthonormal basis for V_j.

Proof Sketch:

$\phi(2^j(t - 2^{-j}k))$

$$\| 2^{j/2} \phi(2^j t - k) \|^2_{L^2}$$

$$= \int_{-\infty}^{\infty} 2^j \phi^2(2^j t - k)\, dt$$

$$= 2^j \int_{2^{-j}k}^{2^{-j}(k+1)} 1^2 \, dt = 2^j \times 2^{-j} = 1$$

You can quickly check for orthogonality (Non overlapping support for 'different' time translates)

$$\left\langle 2^{j/2}\phi(2^j t - i),\ 2^{j/2}\phi(2^j t - k)\right\rangle_{i\neq k} = 0$$

$\boxtimes$

The Haar Wavelet

Motivation : Say, we wanted to isolate a short burst/spike or what we may think of as a high frequency change. We need a tool to isolate the spike $\in V_j$ but not a member of V_{j-1} (Recall : $V_{j-1} \subset V_j$)

Idea : We need to apply 'direct sum' spaces.

i.e., We need to decompose V_j as a sum of V_{j-1} and its 'orthogonal complement'

Intuition & Starting Step

Consider the space V_1. We need to identify a space W_0 (i.e., the orthogonal complement of V_0) with the following properties.

1) $\psi \in V_1 \implies \psi(t) = \sum_{\ell} a_\ell \, \phi(2t - \ell)$

for some $a_\ell \in \mathbb{R}$.

2) ψ is orthogonal to $V_0 \implies \int \psi(t)\, \phi(t-k)\, dt = 0$

$\forall k \in \mathbb{Z}$

Let us see how this can be done

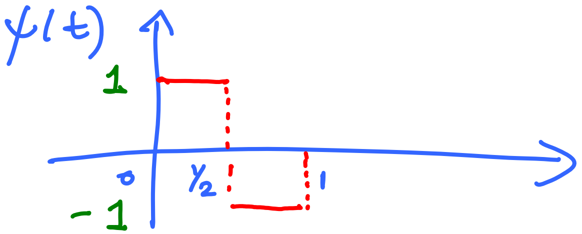
$\psi(t)$ is constructed from box functions of width ½ and its translates.

(Initial case of $k = 0$ from an 2^{nd} cond.)

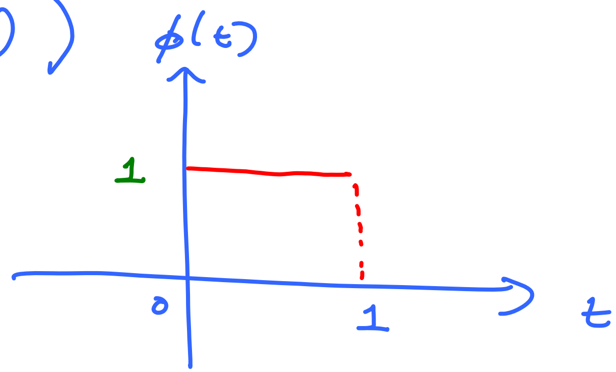
$$\int_{-\infty}^{\infty} \psi(t)\, \phi(t)\, dt = 0$$

$\Rightarrow$ A simple $\psi(t)$ can be of the form

$$\psi(t) = \phi(2t) - \phi\left(2\left(t - 1/2\right)\right)$$



$\perp$



Now, $\psi(t) \perp \phi(t)$

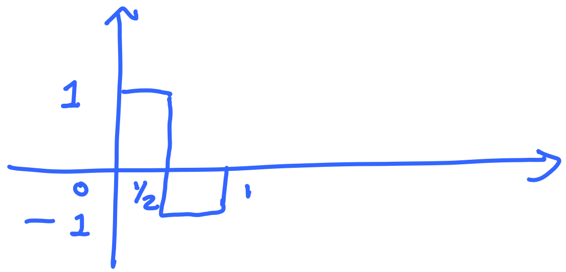
$$V_1 = V_0 \oplus W_0$$

$\therefore \quad \psi(t) \in V_1$

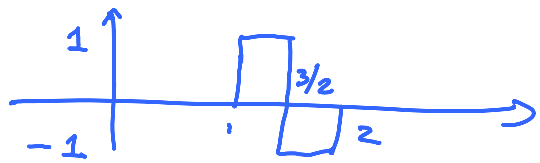
$\psi(t) \in W_0 \quad$ (i.e., $V_0^{\perp}$)

Thus W_0 has all functions of the form

$$\sum_{k \in \mathbb{Z}} a_k \psi(t-k) \qquad a_k \in \mathbb{R}$$



We call $\psi(t)$ as "the wavelet" function!



Let us generalize this into a Theorem

Theorem : Let W_j be the space of all functions / $\sum_{k \in \mathbb{Z}} a_k \psi(2^j t - k)$ $a_k \in \mathbb{R}$.

(1) W_j is the orthogonal complement of V_j in V_{j+1}.

(2) $V_{j+1} = V_j \oplus W_j$

Proof Sketch :

(i) We need to show that every function in W_j is orthogonal to every function in V_j.

Let $f_{W_j} = \sum_{k \in Z} a_k \psi(2^j t - k)$

Let $f_{V_j} \in V_j$. We need to show $\langle f_{W_j}, f_{V_j} \rangle_{L^2} = 0$

From the scaling property $f_{V_j}(t) \in V_j$, then $g(2^{-j} t) \in V_0$

$g \hat{=} f_{V_j}$

Consider

$$\int_{-\infty}^{\infty} \sum_{k \in Z} a_k \, \psi(t-k) \, g(\underbrace{2^{-j} t}) \, dt = 0 \quad \left(\because \psi \text{ is orthogonal to } V_0 \right)$$

CHANGE OF VARIABLE

$t' = 2^{-j} t$

$dt' = 2^{-j} dt$

$$\Rightarrow \quad 2^{j} \int_{-\infty}^{\infty} \underbrace{\sum_{k \in Z} a_k \, \psi(2^{j} t' - k)}_{f(t')} \, g(t') \, dt' = 0$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f(t') \, g(t') \, dt' = 0$$

So, any $g \in V_j$ is orthogonal to $f \in W_j$

(2) When $j = 0$, we showed that any function in V_1 orthogonal to V_0 must be a linear combination of $\{ \psi(t-k), k \in Z \}$

Proceed in a general way for $j \neq 0$ (Part of Home Work)

Lemma : Let $f_1 = \sum_k a_k \phi(2t-k) \in V_1$

$f_1 \perp V_0$ i.e., to each of $\{\phi(t-k)\}_{k \in Z}$

iff $a_1 = -a_0$, $a_3 = -a_2$, ...

Proof : $\phi(t-\ell) = \phi(2t-\ell) + \phi(2t-\ell-1)$ —— ①

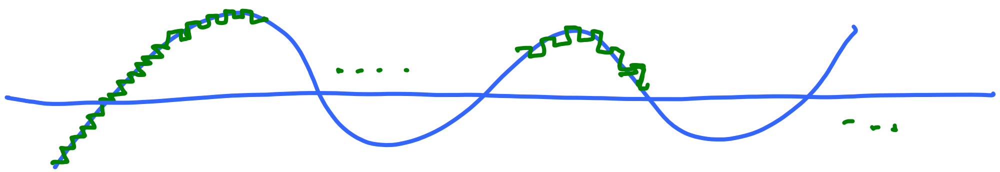
Consider $\left\langle \sum_{k \in Z} a_k \phi(2t-k) , \phi(t-\ell) \right\rangle$ —— ②

Use ① in ②

$$= \sum_{k \in Z} a_k \left[\underbrace{\langle \phi(2t-k), \phi(2t-\ell) \rangle}_{\delta_{k-\ell}} + \underbrace{\langle \phi(2t-\ell-1), \phi(2t-k) \rangle}_{\delta_{k-\ell-1}} \right] = 0$$

$\Rightarrow a_\ell + a_{\ell+1} = 0$ Plug in $\ell = 0, 1, 2, ..$ ☒

Approximating using step functions



GOAL: Approximate a smooth continuous function using step functions.

From the Theorem, we can do successive decompositions of subspaces.

$$
\begin{aligned}
V_j &= W_{j-1} \oplus V_{j-1} \\
&= W_{j-1} \oplus W_{j-2} \oplus V_{j-2} \\
&\;\;\vdots \\
&= W_{j-1} \oplus W_{j-2} \oplus \cdots \oplus W_0 \oplus V_0
\end{aligned}
$$

So, any function f in V_j can be uniquely decomposed as a sum

$$f = w_{j-1} + w_{j-2} + \dots + w_0 + f_0$$

w_l can take care of a short burst of width $\frac{1}{2^{l+1}}$

Theorem : The space $L^2(\mathbb{R})$ can be decomposed into an infinite orthogonal direct sum space

i.e., $L^2(R) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots$

i.e., $f \in L^2(R)$ can be written as

$$f = \underbrace{f_0}_{\in V_0} + \lim_{N \to \infty} \sum_{j=0}^{N} \underbrace{w_j}_{\in W_j}$$

NOTE : There are 2 major points to prove :

a) Any function $f \in L^2(R)$ can be approximated by continuous funct.

b) Any cont. function can be approximated as desired by a step function whose discontinuities are multiples of 2^{-i} for large 'i'.

Haar Decomposition

Intuition: Going back to our motivation, where we wanted to isolate a short burst/spike,

from the Theorem on sub-space decomposition,

$$f_j = f_0 + w_0 + w_1 + \cdots$$

$w_k \in W_k$ with width $2^{-(k+1)}$

Example: Suppose we have a 5 ms spike

$$2^{-7} > .005 > 2^{-8}$$

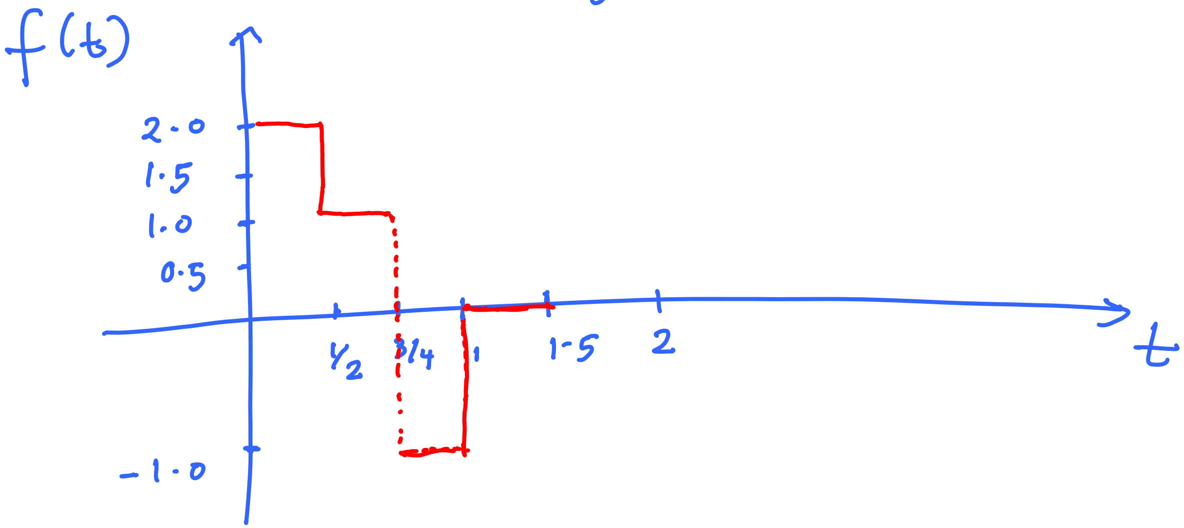
Let us consider the wavelet decomposition process.

Before we begin,

$$\left.\begin{aligned}\phi(2t) &= \left(\psi(t) + \phi(t)\right)/2 \\ \phi(2t-1) &= \left(\phi(t) - \psi(t)\right)/2\end{aligned}\right\} \quad (A)$$

Recall : $\psi(t) = \phi(2t) - \phi(2t-1)$

Consider the following example



OBSERVATIONS

1) Smallest bin is of width $\frac{1}{4}$ time steps

2) We observe this between $[\frac{1}{2}, \frac{3}{4})$ & $[\frac{3}{4}, 1)$

We can describe $f(t)$ over V_2 & express in terms of

$$\{ \phi(2^2 t - l), \ l \in Z \}$$

$$f(t) = 2\,\underline{\phi(4t)} + 2\,\underline{\phi(4t-1)} + \underline{\phi(4t-2)} - \underline{\phi(4t-3)}$$

Let us decompose f into W_1, W_0 & V_0

$$f \in V_2 \Rightarrow f_{V_2} = f_0 + w_0 + w_1$$

From (A), we can generalize

$$\phi(2^j t) = \left(\psi(2^{j-1}t) + \phi(2^{j-1}t)\right)/2$$

$$\phi(2^j t - 1) = \left(\phi(2^{j-1}t) - \psi(2^{j-1}t)\right)/2$$

$$\phi(4t) = \left(\psi(2t) + \phi(2t)\right)/2$$

$$\phi(4t-1) = \left(\phi(2t) - \psi(2t)\right)/2$$

$$\phi(4t-2) = \phi\left(4\left(t-\tfrac{1}{2}\right)\right)$$
$$= \left(\psi\left(2\left(t-\tfrac{1}{2}\right)\right) + \phi\left(2\left(t-\tfrac{1}{2}\right)\right)\right)/2$$

$$\phi(4t-3) = \phi\left(4\left(t-\tfrac{1}{2}\right) - 1\right)$$
$$= \left(\phi\left(2\left(t-\tfrac{1}{2}\right)\right) - \psi\left(2\left(t-\tfrac{1}{2}\right)\right)\right)/2$$

Grouping the terms,

$$f(t) = \psi(2t) + \phi(2t) + \phi(2t) - \psi(2t) + \frac{\big(\psi(2t-1) + \phi(2t-1)\big)}{2} - \frac{\big(\phi(2t-1) - \psi(2t-1)\big)}{2}$$

$$= 2\,\underbrace{\phi(2t)}_{\in V_1} + \underbrace{\psi(2t-1)}_{\in W_1}$$

Decompose $\phi(2t)$ further,

$$\phi(2t) = \frac{\phi(t) + \psi(t)}{2}$$

$$f(t) = \underbrace{\phi(t)}_{\in V_0} + \underbrace{\psi(t)}_{\in W_0} + \underbrace{\psi(2t-1)}_{\in W_1}$$

Let us write things explicitly

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{else} \end{cases}$$

$$\psi(t) = \begin{cases} 1 & 0 \le t < 1/2 \\ -1 & 1/2 \le t < 1 \\ 0 & \text{else} \end{cases}$$

$$\psi(2t-1) = \begin{cases} 1 & 1/2 \le t < 3/4 \\ -1 & 3/4 \le t < 1 \\ 0 & \text{else} \end{cases}$$

Add all the components

$$\phi(t) + \psi(t) + \psi(2t-1) = \begin{cases} 2 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < \frac{3}{4} \\ -1 & \frac{3}{4} \le t < 1 \\ 0 & \text{else} \end{cases}$$

We are home

Let us recall the two formulae we used for wavelet recursions

$$\left.\begin{aligned} \phi(2^{j} t) &= \frac{1}{2}\left[\psi(2^{j-1} t) + \phi(2^{j-1} t)\right] \\ \phi(2^{j} t - 1) &= \frac{1}{2}\left[\phi(2^{j-1} t) - \psi(2^{j-1} t)\right] \end{aligned}\right\} \text{—— } B$$

We will try to derive a general procedure for wavelet decomposition of a signal from an arbitrary scale 'j'.

Since $\{ \phi(2^j t - k),\ k \in \mathbb{Z} \}$ form an orthonormal basis for scale 'j',

$$f_j(t) = \sum_{k \in \mathbb{Z}} a_k \, \phi(2^j t - k) \quad \text{———} \quad (C)$$

Replace 't' by $t - k\, 2^{-(j-1)}$ in (B)

$$\left.\begin{aligned} \phi(2^j t - 2k) &= \frac{1}{2}\left[\psi(2^{j-1}t - k) + \phi(2^{j-1}t - k)\right] \\ \phi(2^j t - 2k - 1) &= \frac{1}{2}\left[\phi(2^{j-1}t - k) - \psi(2^{j-1}t - k)\right] \end{aligned}\right\} \text{—— } (1)$$

Using (1) in (6), we have the following:

First, let us split (6) into even and odd terms and then apply (1).

$$f_j(t) = \sum_{k \in Z} a_{2k}\, \phi(2^j t - 2k) + \sum_{k \in Z} a_{2k+1}\, \phi(2^j t - 2k - 1) \quad (2)$$

Apply (1) in (2), we have the following,

$$f_j(t) = \frac{1}{2}\sum_{k\in Z} a_{2k}\left[\phi(2^{j-1}t-k) + \psi(2^{j-1}t-k)\right] + \frac{1}{2}\sum_{k\in Z} a_{2k+1}\left[\phi(2^{j-1}t-k) - \psi(2^{j-1}t-k)\right]$$

$$= \sum_{k\in Z} \underbrace{\left(\frac{a_{2k}-a_{2k+1}}{2}\right)}\psi(2^{j-1}t-k) + \sum_{k\in Z}\underbrace{\left(\frac{a_{2k}+a_{2k+1}}{2}\right)}\phi(2^{j-1}t-k) \tag{3}$$

$$f_j(t) = w_{j-1}(t) + f_{j-1}(t)$$

where $w_{j-1}(t)$ is the W_{j-1} component of $f_j(t)$ and is a linear span of $\{\psi(2^{j-1}t - k),\ k \in \mathbb{Z}\}$

and $f_{j-1}(t)$ is the V_{j-1} component of $f_j(t)$ in the linear span of $\{\phi(2^{j-1}t - k),\ k \in \mathbb{Z}\}$

Let us be slightly more precise by introducing a superscript 'j' on a_ks.

$$f_j(t) = \sum_{k \in Z} a_k^{(j)} \phi(2^j t - k) \quad \in V_j$$

$$f_j(t) = w_{j-1}(t) + f_{j-1}(t)$$

$$w_{j-1}(t) = \sum_{k \in Z} b_k^{(j-1)} \psi(2^{j-1} t - k) \quad \in W_{j-1}$$

where $b_k^{(j-1)} = \dfrac{a_{2k}^{(j)} - a_{2k+1}^{(j)}}{2}$

$|||^{ly}$

$$f_{j-1}(t) = \sum_{k \in \mathbb{Z}} a_k^{(j-1)} \phi(2^{j-1} t - k) \in V_{j-1}$$

where

$$a_k^{(j-1)} = \frac{a_{2k}^{(j)} + a_{2k+1}^{(j)}}{2}$$

We can proceed computing the recursions for $j-1, j-2, \cdots 0$

$$f_j(t) = f_0(t) + \sum_{i=0}^{j-1} w_i(t)$$

This gives us the 'Forward decomposition' procedure!