The Idea of Sampling

Let $X(\omega)$ be the spectrum of $x(t)$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

If $X(\omega)$ is assumed to be zero outside the band $|\omega| < 2\pi B$,

$$x(t) = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} X(\omega)\, e^{j\omega t}\, d\omega$$

Let $t = \frac{n}{2B}$

$$x\left(\frac{n}{2B}\right) = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} X(\omega)\, e^{j\omega \frac{n}{2B}}\, d\omega$$

The L.H.S has $x(t)$ at the sampling points.

The integral on the right is essentially the n^{th} coefft in the Fourier series expansion of $X(\omega)$ over the interval $[-B, B]$ as a fundamental period.

$\left\{ x\left(\frac{n}{2B}\right) \right\}$ determine the F. coeffts in the series expansion of $X(\omega)$

Since $X(\omega)$ is zero for frequencies $> B$ &

$X(\omega)$ is determined fully if the coeffts are known,

the samples $\left\{ x\left(\frac{n}{2B}\right) \right\}$ determine $x(t)$ completely.

Problem : How do we 're' construct $x(t)$ from the samples ?

Let us start with the Dirac comb function

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{k=-\infty}^{\infty} c_k \, e^{j 2\pi \frac{k}{T} t}$$

$c_k = \frac{1}{T}$

Periodic $\Rightarrow$ F. series representation

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} \, e^{j 2\pi \frac{k}{T} t} \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k/T)$$

Consider

$$\sum_{k=-\infty}^{\infty} F(k) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi k t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \underbrace{\sum_{k=-\infty}^{\infty} e^{-j2\pi k t}}_{\sum_{n=-\infty}^{\infty} \delta(t-n)} dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, \delta(t-n)\, dt = \sum_{n=-\infty}^{\infty} f(n)$$

|||ly Consider

$$\sum_{k=-\infty}^{\infty} S\left(\omega + 2\pi k/T\right) = \sum_{k=-\infty}^{\infty} \mathcal{F}\left(s(t)\, e^{-j2\pi \frac{k}{T} t}\right)$$

$$= \mathcal{F}\left(s(t) \underbrace{\sum_{k=-\infty}^{\infty} e^{-j2\pi \frac{k}{T} t}}_{T \sum_{n=-\infty}^{\infty} \delta(t-nT)}\right)$$

$$= \mathcal{F}\left(s(t)\; T \sum_{n=-\infty}^{\infty} \delta(t-nT)\right)$$

$$= \mathcal{F}\left(\sum_{n=-\infty}^{\infty} s(nT)\; T\; \delta(t-nT)\right)$$

$$= \sum_{n=-\infty}^{\infty} T \cdot s(nT)\, \mathcal{F}\left(\delta(t-nT)\right)$$

$$= \sum_{n=-\infty}^{\infty} T \cdot s(nT)\, e^{-j2\pi nTf}$$

Sampling process converts a continuous time signal into a signal of discrete time.

Sampling Theorem :

If a signal $s(t)$ contains no frequencies outside B Hz, it is completely determined by its values at a sequence of points spaced $< \frac{1}{2B}$ seconds apart.

Let us consider the periodic summation of $S(f)$

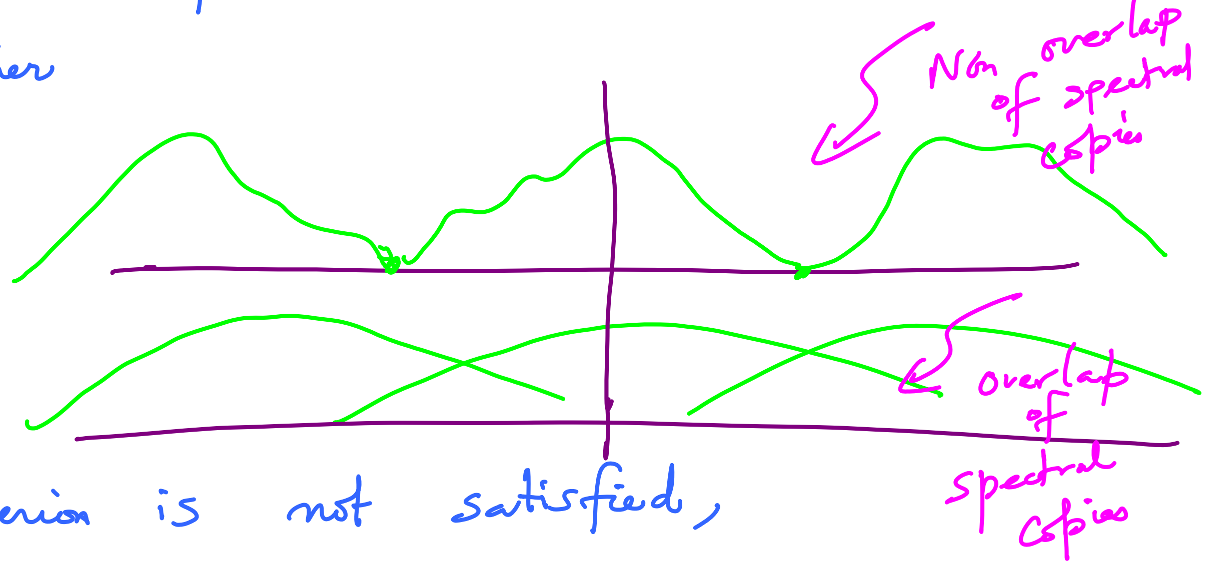
$$S_{\text{periodic sum}}(f) = \sum_{k=-\infty}^{\infty} S(f - k f_s)$$

where $f_s = \frac{1}{T}$ "sampling rate".

$$= \sum_{n=-\infty}^{\infty} T\, s(nT)\, e^{-j2\pi nTf}$$

Copies of $S(f)$ in multiples of f_s, translated are added!

For bandlimited signals i.e., $X(f) = 0$; $|f| \geq B$ & sufficiently large f_s, it is possible for the copies to be distinct from each other



If the Nyquist criterion is not satisfied,

adjacent copies overlap $\Longrightarrow$ aliasing effect

Derive the interpolation formula

$S_{\text{periodic sum}}(f)$ can be used to recover $S(f)$

i.e., with $k = 0$

$$S(f) = H(f)\, S_{\text{periodic sum}}(f)$$

$$H(f) \triangleq \begin{cases} 1 & |f| < B \\ 0 & |f| > f_s - B \end{cases}$$

CRITICAL POINT IS AT $B = f_s/2$ i.e., @ Nyquist

Use the fact

$$H(f) = \text{rect}\left(f/f_s\right) = \begin{cases} 1 & |f| < \frac{f_s}{2} \\ 0 & |f| > \frac{f_s}{2} \end{cases}$$

$$S(f) = \text{rect}\left(f/f_s\right) S_{\text{periodic sum}}(f)$$

$$= \text{rect}\left(Tf\right) \sum_{n=-\infty}^{\infty} T\, s(nT)\, e^{-j2\pi nTf}$$

$$= \sum_{n=-\infty}^{\infty} s(nT) \underbrace{T \cdot \text{rect}(Tf)\, e^{-j2\pi nTf}}_{\mathcal{F}\left(\text{sinc}\left(\frac{t-nT}{T}\right)\right)}$$

Taking inverse F.T on f.s.

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT)\, \underbrace{\mathrm{sinc}\left(\frac{t-nT}{T}\right)}_{\text{Sinc Interpolator}}$$

Other Considerations

A) The sampling theory can be generalized when samples are not taken equally spaced in time.

Henry Landau on non base band, non uniform sampling

B) Recent well developed theory on Compressed sensing

Idea: This allows for full reconstruction with Sub Nyquist Sampling rate for signals that are sparse i.e., compressible

Low overall bandwidth but freq. locations are unknown rather than everything in one band

Basics of multirate systems

Motivation

1) Sampling rate converters
2) Oversampled systems

Studio work	:	48 KHz
Digital tape/ CDs	:	44.1 KHz
Broadcasting	:	32 KHz

How can I cater to a digital system working at diff. sampling rates with the same analog signal?

Naive Solution: Use ADCs, DACs at every point where a conversion is needed!

Basic multirate operations

1) Decimation

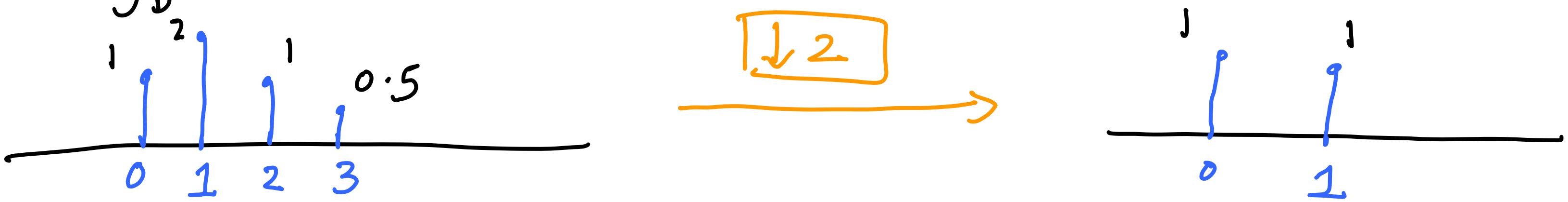
2) Expansion

M- fold decimator

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y_D[n]$$

a. k. a "Compressor"

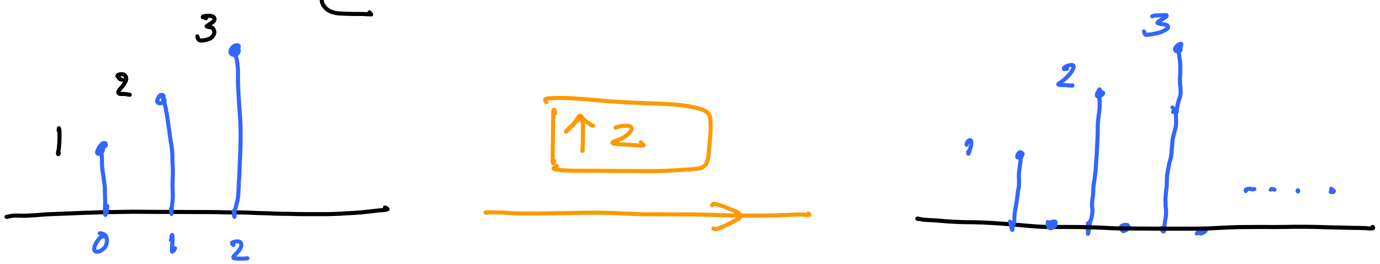
$$y_D[n] = x[Mn]$$



L fold expander (Upsampler)

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow y_E(n)$$

$$y_E(n) = \begin{cases} x(n/L) & \text{if } n \text{ is an integer multiple of } L \\ 0 & \text{else} \end{cases}$$



Exercise : Verify that decimators & expanders are linear but 'time varying'

LTV systems

Frequency domain effects of the decimator

$$Y_D(z) = \sum_{n=-\infty}^{\infty} y_D[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(Mn)\, z^{-n}$$

Let us define a sequence

$$x_1(n) = \begin{cases} x(n) & n \text{ is a multiple of } M \\ 0 & \text{else} \end{cases}$$

$$Y_D(z) = \sum_{n=-\infty}^{\infty} x_1(Mn)\, z^{-n} = \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k/M} \quad (Mn = k)$$

$$x_1(n) = c_M(n)\, x(n)$$

where $c_M(n) = \begin{cases} 1 & n \text{ is a multiple of } M \\ 0 & \text{else} \end{cases}$

'COMB SEQUENCE'

$$c_M(n) = \frac{1}{M} \sum_{k=0}^{M-1} \omega_M^{-kn}$$

where $\omega_M = e^{-j2\pi/M}$ (M^{th} root of unity)

$$X_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x(n)\, \omega^{-kn} z^{-n}$$

$$X_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} \underbrace{\sum_{n=-\infty}^{\infty} x(n) \left(z\, \omega^{k}\right)^{-n}}_{X(z\omega^k)}$$

$z^M - 1$

$(z-\omega_0)(z-\omega_1)\ldots(z-\omega_{M-1})$

$\Rightarrow$ coefft of z^{M-1} is 0

Hence $\sum$ roots of unity $= 0$

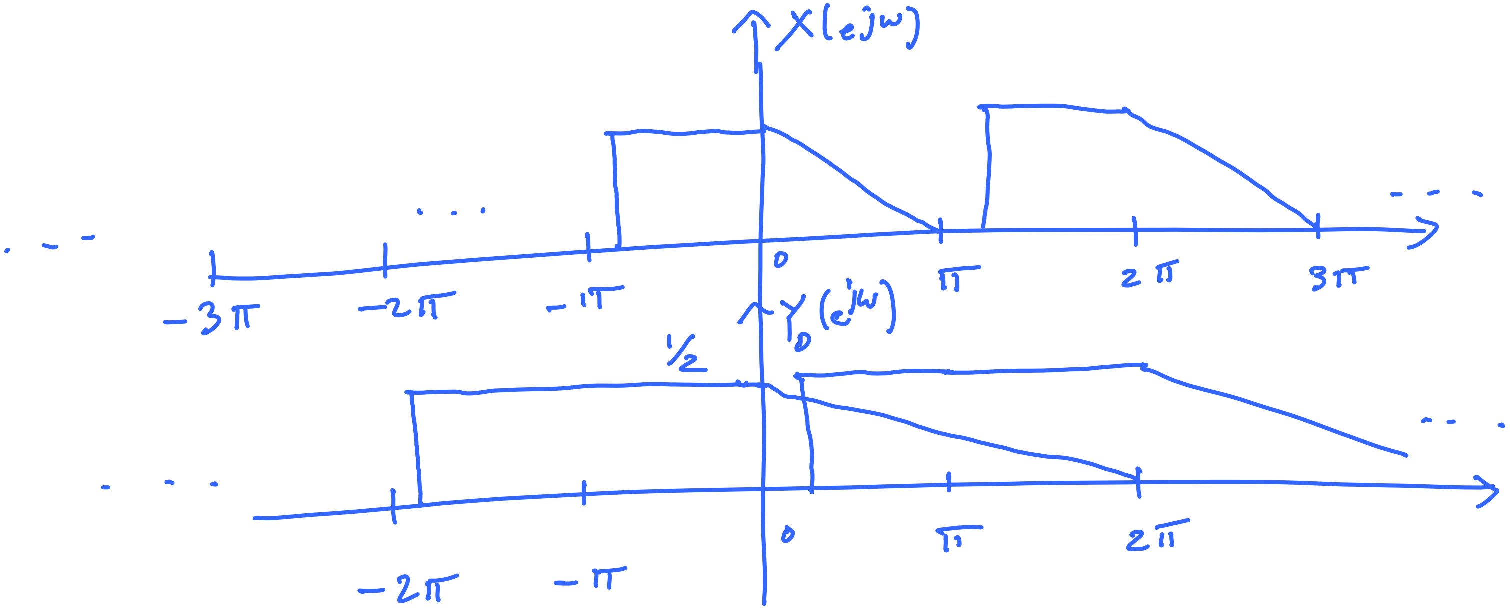
$$Y_D(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} \, W_M^{k}\right) \qquad \left(\because \; Y_D(z) = X_1\left(z^{\frac{1}{M}}\right) \right)$$

$$W_M \triangleq e^{-j\frac{2\pi}{M}}$$

$$Y_D(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j(\omega - 2\pi k)/M}\right)$$

4 diff. operations

1) Stretch $X(e^{j\omega})$ by a factor M to obtain $X(e^{j\omega/M})$
2) Create copies i.e., $M-1$ copies of this 'stretched signal' by shifting it uniformly in successions of 2π.
3) Add the shifted versions to the 'unshifted' stretched version
4) Scale by M



$\downarrow 2$

Frequency domain analysis of expansion

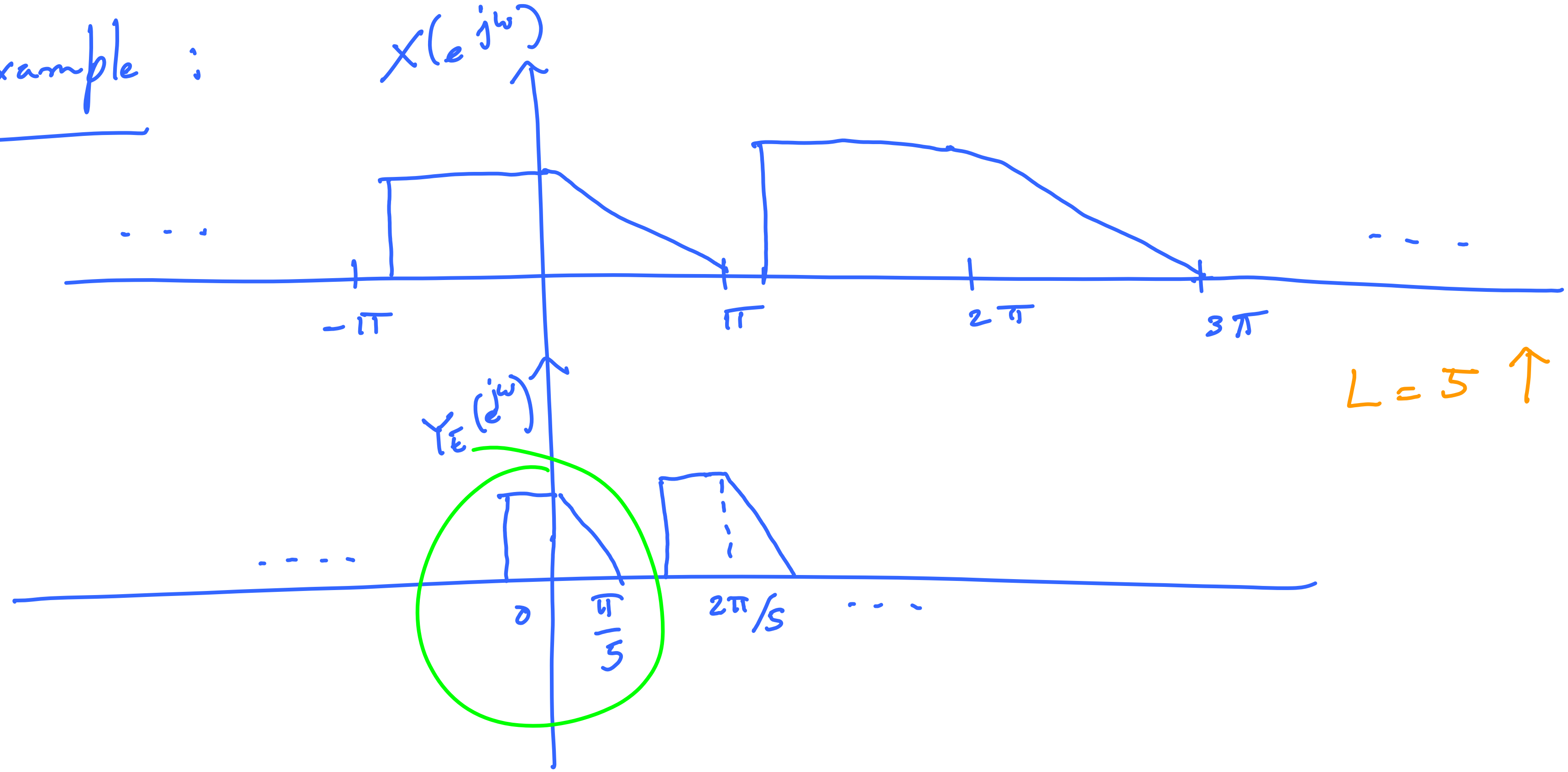
$$Y_E(z) = \sum_{n=-\infty}^{\infty} y_E(n)\, z^{-n}$$

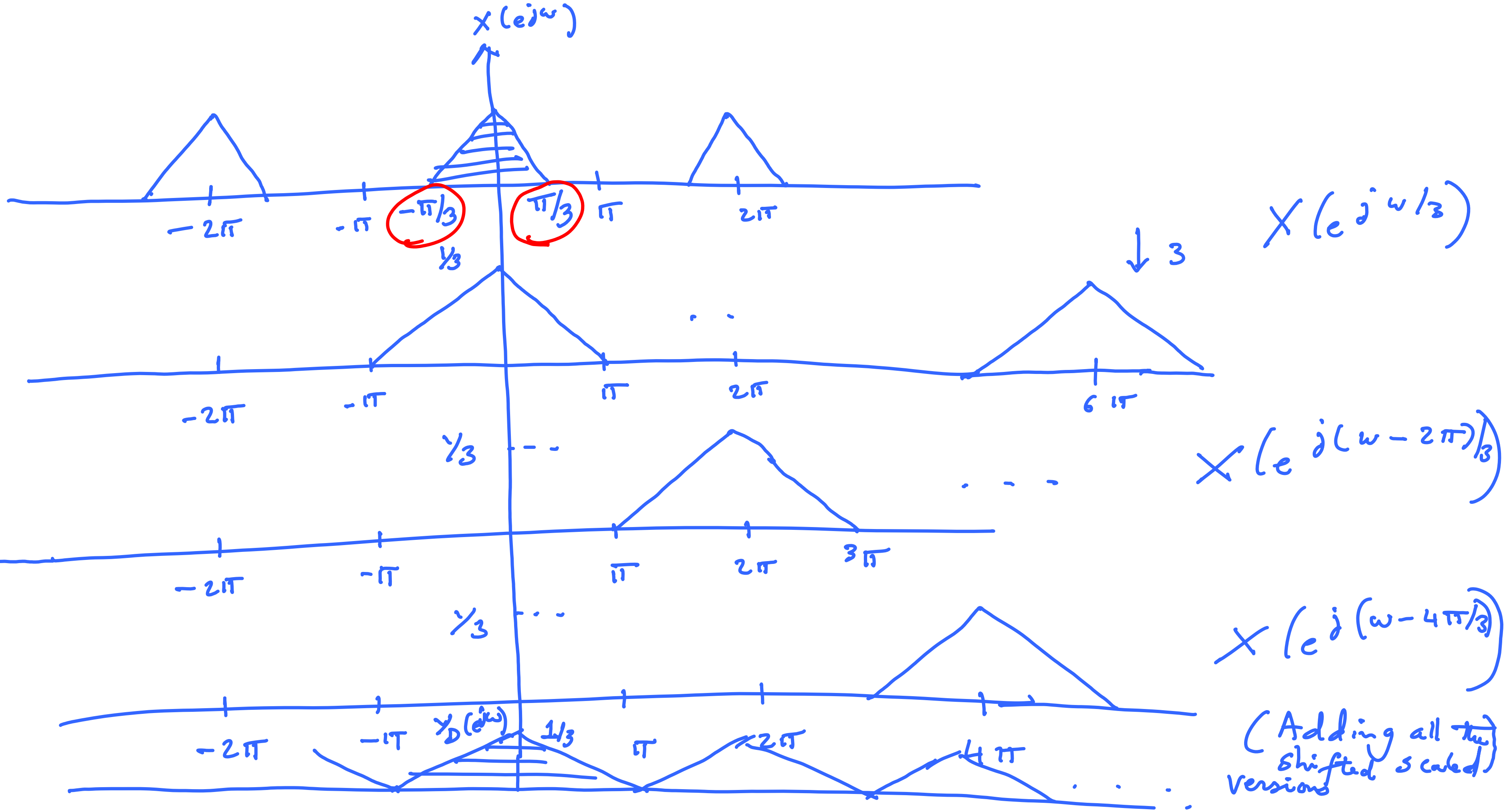
$$= \sum_{n \text{. multiple of } L} y_E(n)\, z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} y_E(kL)\, z^{-kL} = \sum_{k=-\infty}^{\infty} x(k)\, z^{-kL}$$

$$= X(z^L)$$

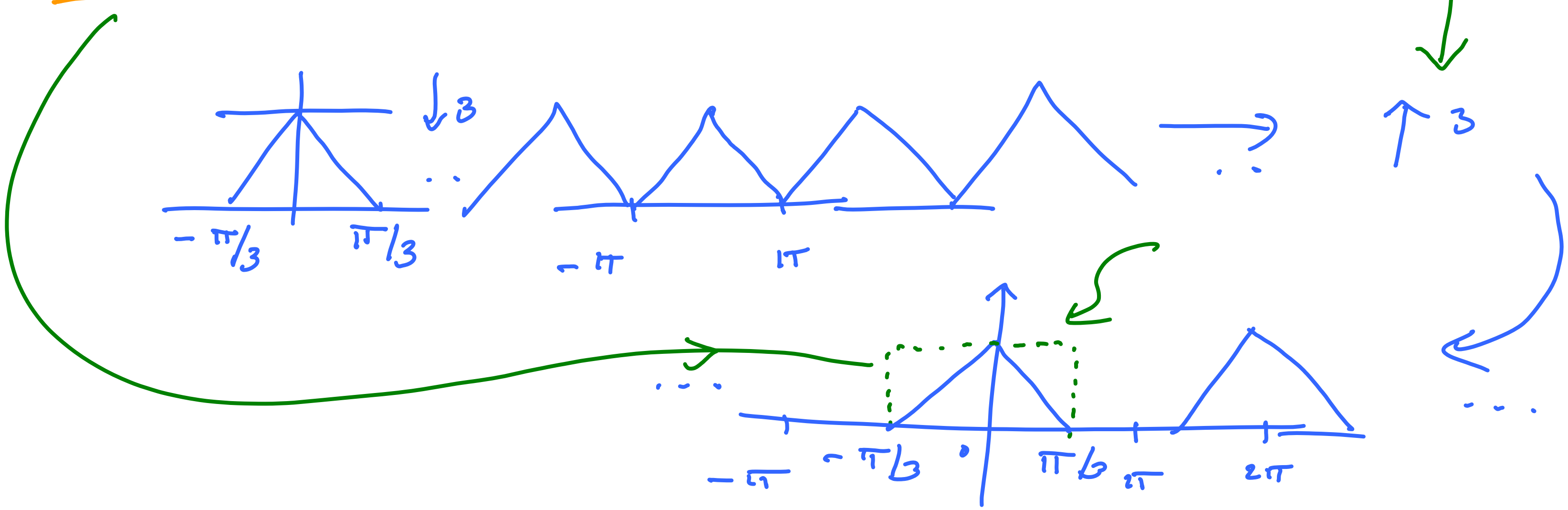
Example :





$X(e^{j\omega})$
-2π
$-\pi$
$-\pi/3$
$\pi/3$
π
2π
$\downarrow 3$
$X(e^{j\omega/3})$
1/3
6π
$X(e^{j(\omega-2\pi)/3})$
3π
$X(e^{j(\omega-4\pi)/3})$
4π
$Y_D(e^{j\omega})$
(Adding all the shifted scaled versions)

Avoid Aliasing: To avoid aliasing, $x(n)$ is a low pass signal band limited to the region $|w| < \frac{\pi}{M}$. To recover $x(n)$ we expand the decimated version followed by filtering.



If the spectrum $X(e^{j\omega})$ is zero everywhere in $0 \le \omega < 2\pi$ except in $\omega_1 < \omega < \omega_1 + 2\pi/M$ for some ω_1, then there is no overlap between any pair of terms within

$$Y_D(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} w^{k}\right)$$

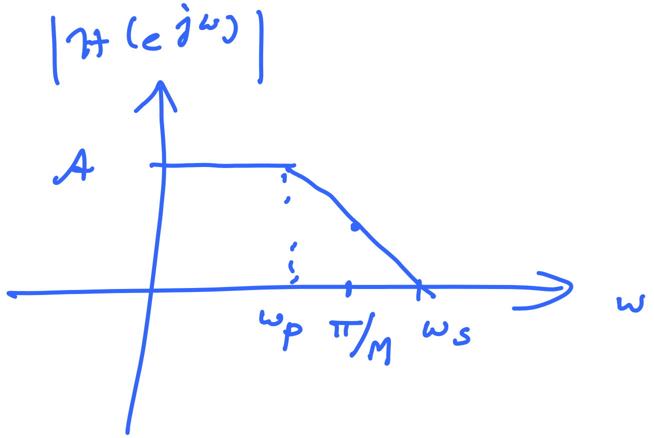
Decimation & Interpolation Filters

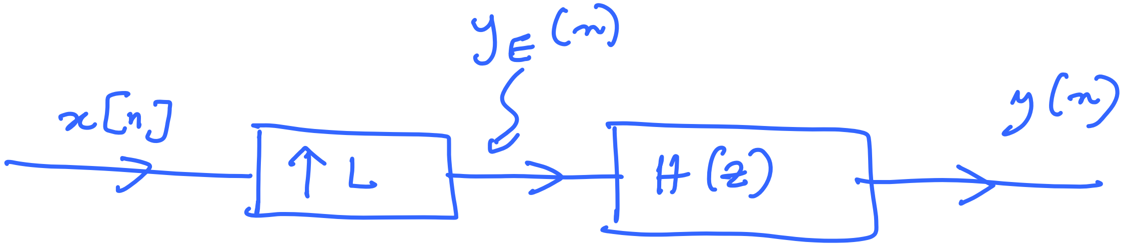
In most applications, decimation is preceded by a low pass digital filter a.k.a. '<u>decimation filter</u>'. The filter ensures that the signal being decimated is band limited.

The exact band edges of the filter depend on how much aliasing is permitted.

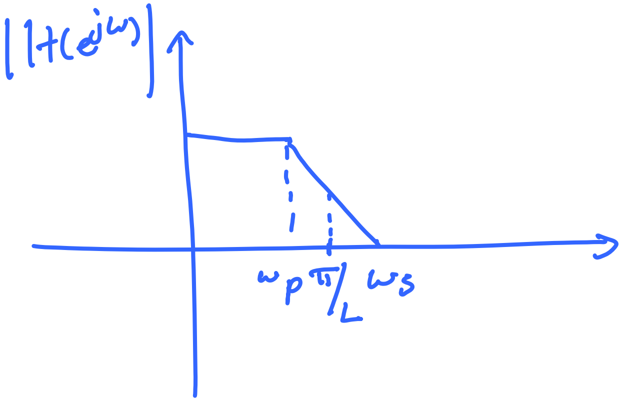
$$x[n] \longrightarrow \boxed{H(z)} \longrightarrow \boxed{\downarrow M} \longrightarrow y(n)$$

Filtering followed by decimation



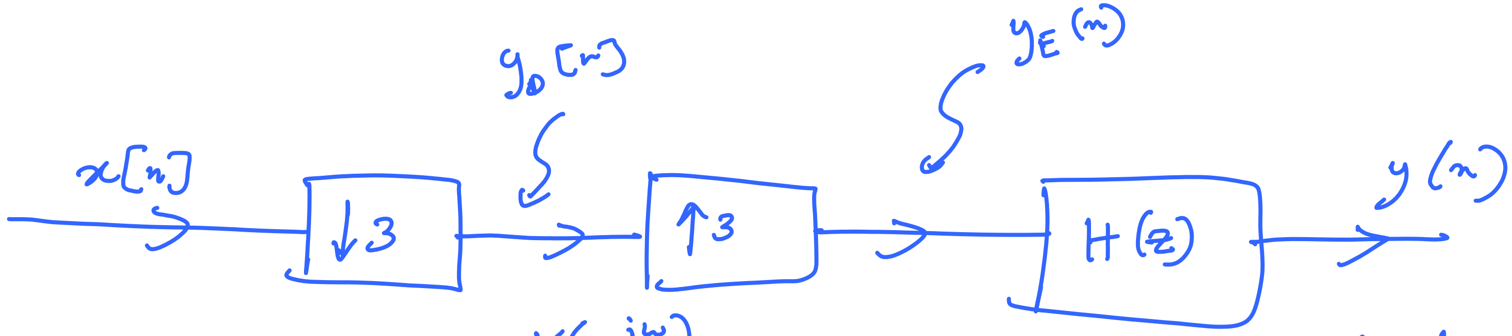


x[n]
↑ L
y_E(n)
H(z)
y(n)

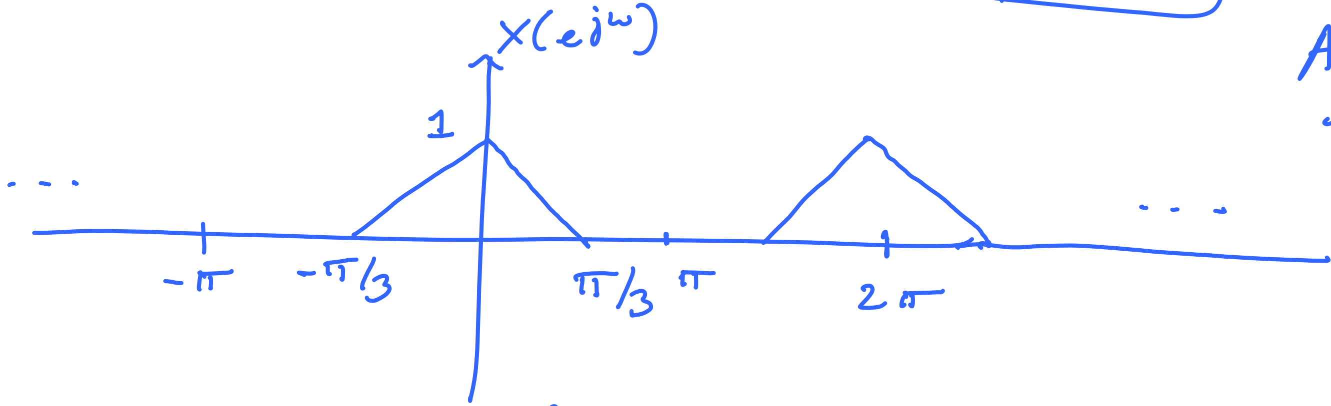


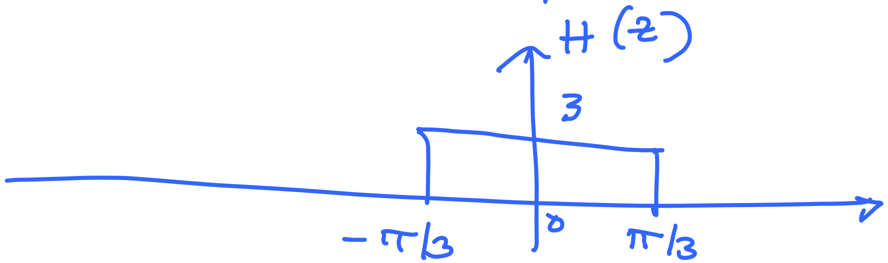
|H(e^{jω})|
ω_p
π/L
ω_s

Exercise :



Analyze the spectrum of y[n]





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