Some notions and properties on Convergence of functions

Pointwise Convergence

A sequence of functions $\{f_n\}_{n \geq 0}$ defined on a set S converges pointwise to a function f defined on S if

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{holds } \forall\, x \in S.$$

In other words,

f_n converges pointwise to f on S if $\forall x \in S$ and $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ (set of natural nos) $/ \forall n \geq N$

$$|f_n(x) - f(x)| < \varepsilon.$$

Here N depends on both ε and x.

Example : Consider the graph of a continuous function $f_n(x) = x^n$ over $[-1, 1]$

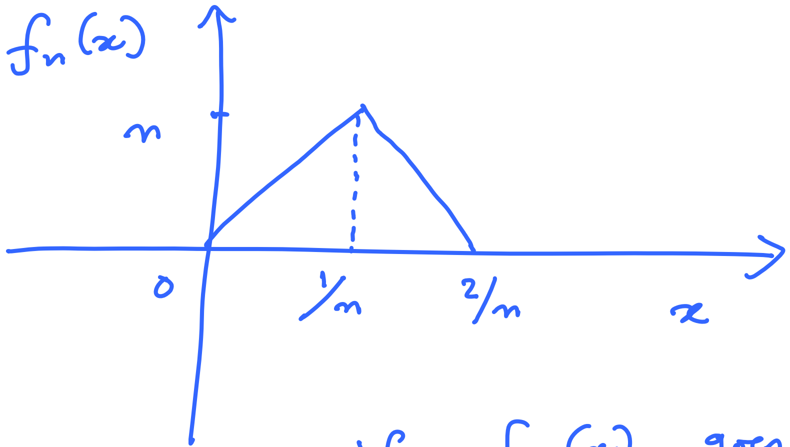
On this set

$$f(x) = \begin{cases} 0 & -1 < x < 1 \\ 1 & x = 1 \end{cases}$$

The limiting function i.e., $f(x)$ is discontinuous

$\Rightarrow$ Point wise limit of a cont. function need not be continuous.

Example : Consider a sequence of piecewise linear functions "tent" functions



$$f_n(x) = \begin{cases} n^2 x & 0 \le x < \frac{1}{n} \\ 2n - n^2 x & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \frac{2}{n} \le x \le 1 \end{cases}$$

Let us examine if $f_n(x)$ goes to zero pointwise on $[0,1]$

If $x \in (0,1]$, $f_n(x) = 0 \quad \forall\, x > \frac{2}{n}$

If $x = 0$, $f_n(0) = 0 \quad \forall\, n$

Observe : $\int_0^1 f_n(x)\,dx = 1 \quad \forall\, n \quad \left(\because \frac{1}{2} \cdot \frac{2}{n} \cdot n = 1 \right)$

$$1 = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \ne \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0 \cdot dx = 0$$

UNIFORM CONVERGENCE

A sequence $\{f_n\}_{n\geq 0}$ defined on a set S converges uniformly to a function f if for every $\varepsilon > 0$ $\exists\, N \in \mathbb{N} \,/\, n \geq N$ $|f_n(x) - f(x)| < \varepsilon$ holds $\forall\, x \in S$

i.e., $\forall\, \varepsilon > 0 \; \exists\, N \in \mathbb{N} \,/\, \forall\, n \geq N, \; \forall\, x \in S$ $|f_n(x) - f(x)| < \varepsilon$ i.e., N depends on ε but not on x

NOTE: If f_n converges to f uniformly on S, then f_n converges to f pointwise as well

Example: Let us examine if

$$\left\{ f_n := \frac{nx^2+1}{nx+1} \right\}$$ is uniformly convergent over $[1, 3]$

First, let us take the "pointwise limit"

$$\lim_{n\to\infty} \frac{nx^2+1}{nx+1} = \lim_{n\to\infty} \frac{x^2+1/n}{x+1/n} = x$$

i.e., f_n converges to x pointwise over $[1, 3]$

Let us examine uniform convergence.

Consider $\left| f_n(x) - f(x) \right| = \left| f_n(x) - x \right|$

$$\left| \frac{nx^2+1}{nx+1} - x \right| = \left| \frac{1-x}{nx+1} \right| \leq \frac{1+|x|}{nx+1}$$

Over $[1, 3]$, $\frac{1+|x|}{nx+1}$ can be upper bounded

to $\frac{4}{n+1}$ $\forall x \in [1, 3]$

If $\varepsilon > 0$ is chosen $\exists$ N / $n \geq N$

$\frac{4}{n+1} < \varepsilon$

$\Rightarrow n \geq N$

$$\left| f_n(x) - f(x) \right| < \varepsilon \quad \forall x \in [1, 3]$$

This proves <u>UNIFORM CONVERGENCE</u>

Applications

We know that the Fourier series for a 2π periodic function is

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

In functional series form, the above can be written as

$$\sum_{k=0}^{\infty} S_k(x) = \lim_{N \to \infty} \sum_{k=0}^{N} S_k(x)$$

(if limit exists)

It is also obvious that different values of x can give different limits.

Before we go further, let us recall the basic defns of supremum & infimum.

Defn : Let $S \subset \mathbb{R}$. The supremum of S denoted by $\sup S$ is the smallest number $a \in \mathbb{R} \,/\, x \leq a \;\forall\; x \in S$

$$\sup S = \min \left\{ a \in \mathbb{R} : x \leq a \;\forall\; x \in S \right\}$$

$|||^{ly}$ infimum of S is the largest number $b \in \mathbb{R}$ / $x \geq b \ \forall x \in S$

$$\inf S = \max \{ b \in \mathbb{R} : x \geq b \ \forall x \in S \}$$

Let us define for a real valued function on a non empty set S, the supremum on the set S

$$\|f\|_S = \sup_{x \in S} |f(x)|$$

If f is a bounded function on S then

$$\sup_{x \in S} |f(x)| = \sup \{ |f(x)| : x \in S \}$$

exists

Observe that $|f(x)| \le \|f\|_S \quad \forall x \in S$

UNIFORM CONVERGENCE IMPLIES POINT WISE CONVERGENCE

From uniform convergence

$$\left| f_n(x) - f(x) \right| \leq \sup_{x \in S} \left| f_n(x) - f(x) \right|$$

$$= \| f_n - f \|_S$$

So that $f_n \xrightarrow[n \to \infty]{} f$ uniformly on S

$$\Longrightarrow \left| f_n(x) - f(x) \right| \xrightarrow[n \to \infty]{} 0 \text{ for each } x \in S$$

$$\Longrightarrow f_n \xrightarrow[n \to \infty]{} f \text{ point wise on } S$$

Theorem: Suppose $\{f_n(x)\}_{n\geq 0}$ is a sequence of continuous functions on an interval S. Suppose $f_n(x)$ converges uniformly to $f(x)$ on S. Then the limit function $f(x)$ is also continuous.

PROOF: We need to establish $f(x) \xrightarrow[x \to a]{} f(a)$ $\forall\, x, a \in S$.

Let us start with

$$\left| f(x) - f(a) \right|$$

For any $n \geq 0$ i.e., $n = 0, 1, 2, \ldots$

$$\left| f(x) - f(a) \right| = \left| \left(f(x) - f_n(x)\right) + \left(f_n(x) - f_n(a)\right) + \left(f_n(a) - f(a)\right) \right|$$

$$\leq \left| f(x) - f_n(x) \right| + \left| f_n(x) - f_n(a) \right| + \left| f_n(a) - f(a) \right| \quad (\because \text{TRIANGLE INEQUALITY})$$

$$|f(x)-f(a)| \leq 2\|f-f_n\|_S + |f_n(a)-f(a)|$$

$$\left(\because \quad \begin{array}{l} |f(x)-f_n(x)| \leq \|f-f_n\|_S \\ |f(a)-f_n(a)| \leq \|f-f_n\|_S \end{array} \right)$$

Choose a positive number $\varepsilon > 0$ arbitrary small

$$\|f-f_n\|_S \xrightarrow[n\to\infty]{} 0$$

$\therefore \exists N > 0$ for which $\|f_n - f\|_S < \frac{\varepsilon}{3}$

$\forall n \geq N$

Now, $f_N(x)$ is continuous. So, for any choice of $\varepsilon > 0$, there is an interval centered around 'a' so that

$$\left| f_N(x) - f_N(a) \right| < \varepsilon/3$$

whenever $x \in$ that interval

Formally, since $f_N(x) \xrightarrow[x \to a]{} f_N(a)$

$\forall \varepsilon > 0$, there is a corresponding $\delta > 0$ so that $\left| f_N(x) - f_N(a) \right| < \frac{\varepsilon}{3}$ whenever $|x - a| < \delta$

Thus,

$$|f(x) - f(a)| \le 2\,\|f - f_N\|_S + \underbrace{|f_N(x) - f_N(a)|}$$

$$< 2\,\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$$\forall\ |x - a| < \delta$$

$$\Longrightarrow\ f(x) \xrightarrow[x \to a]{} f(a)$$

$\Longrightarrow$ $f(x)$ is <u>CONTINUOUS</u>

Quick Test towards uniform Convergence

If $\{f_n\}_{n\geq 0}$ is a seq. of continuous functions, $f_n(x)$ converges pointwise to $f(x)$; However if the limit function $f(x)$ is '<u>NOT</u>' continuous

$\Rightarrow$ $f_n(x)$ does not converge uniformly to $f(x)$

Theorem: Suppose $\{f_n(x)\}_{n\geq 0}$ is a seq. of continuous functions which converges uniformly to a cont. function $f(x)$ on a bounded interval $[a, b]$. We have

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx$$

$$= \int_a^b f(x)\, dx$$

PROOF :

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right|$$

$$= \left| \int_a^b \big(f_n(x) - f(x)\big)\,dx \right|$$

$$\leq \int_a^b \left| \big(f_n(x) - f(x)\big) \right| dx$$

$$\leq \int_a^b \| f_n - f \|_S \,dx = \| f_n - f \|_S \int_a^b 1 \cdot dx$$

$$= \| f_n - f \|_S (b - a) \xrightarrow[n \to \infty]{} 0$$

Fourier Series : Properties and notions of convergence

Fourier Series : Over the interval $-\pi \le x \le \pi$

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \; ; \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

a_i's & b_i's are the Fourier coeffts.

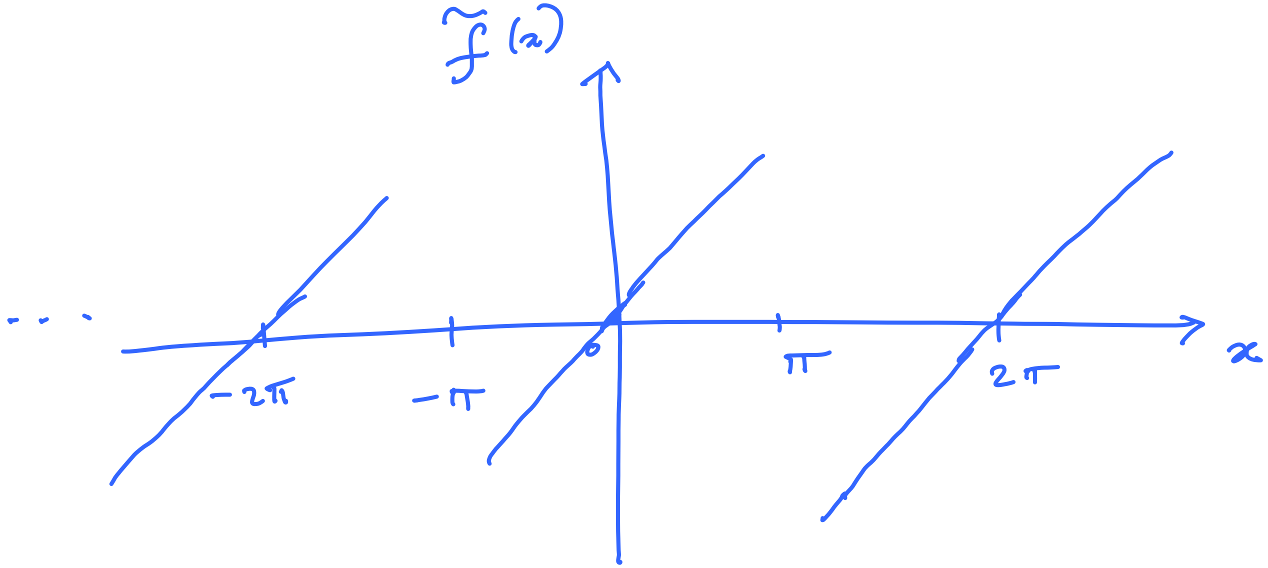
Consider the function $f(x) = x$ on $-\pi \leq x \leq \pi$

$f(x) = -f(-x)$ (odd function)

$$\therefore \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx)\, dx$$

$$F(x) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx)$$

For this example, $f(x)$ is <u>not</u> 2π periodic! Let us form a function $\tilde{f}$ which is a periodic extension of f.



k = odd multiples

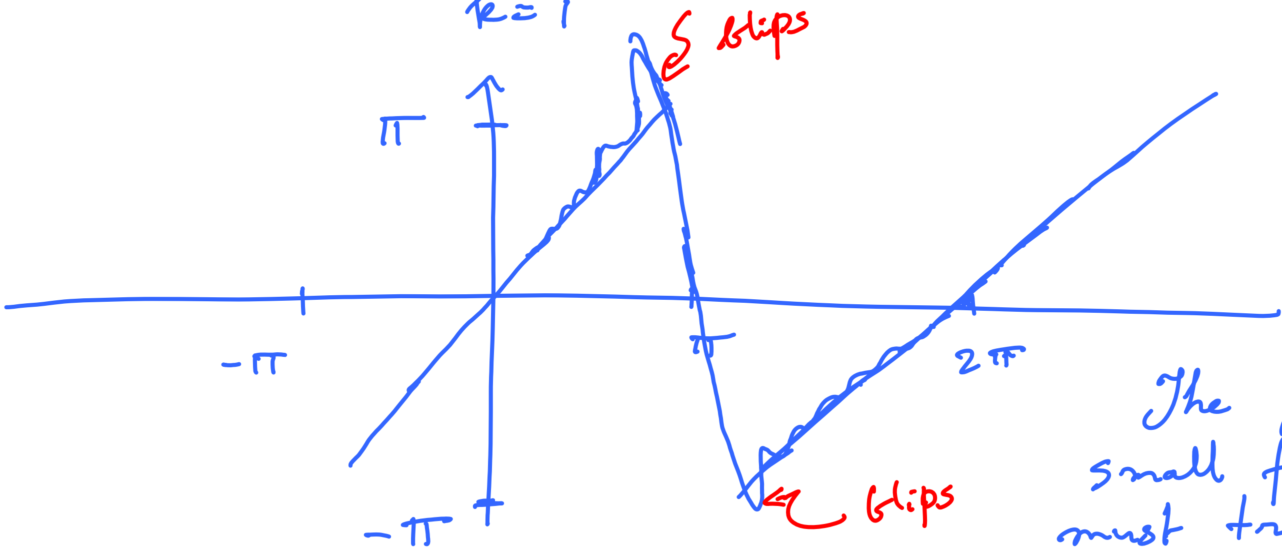
$F(x)$ converges to $\tilde{f}(x)$ at points where $\tilde{f}$ is continuous.

However, we have discontinuity points @ $\pm k\pi$, $k \in \mathbb{Z}$; k odd

At these points $F(x)$ will converge to $\dfrac{f(k\pi^+) + f(k\pi^-)}{2}$

Consider the finite sum

$$S_N(x) = \sum_{k=1}^{N} \frac{2(-1)^{k+1}}{k} \sin(kx)$$



Accuracy of the blips around the discontinuity points gets 'worse'!

The graph of $S_N(x)$ for small finite 'N' say $N=10$ must travel from π to $-\pi$ in a "very short interval".

The blips occurring just before and after the points of discontinuity is called 'GIBBS EFFECT'!

1) The height of the blip is $\sim$ same for large N

2) The width gets smaller as N gets larger.

Exercise :

1) Plot $S_N(x)$ for $N = 10, 100, 1000, \ldots$ Observe Gibbs effect.

2) Investigate for a saw tooth wave

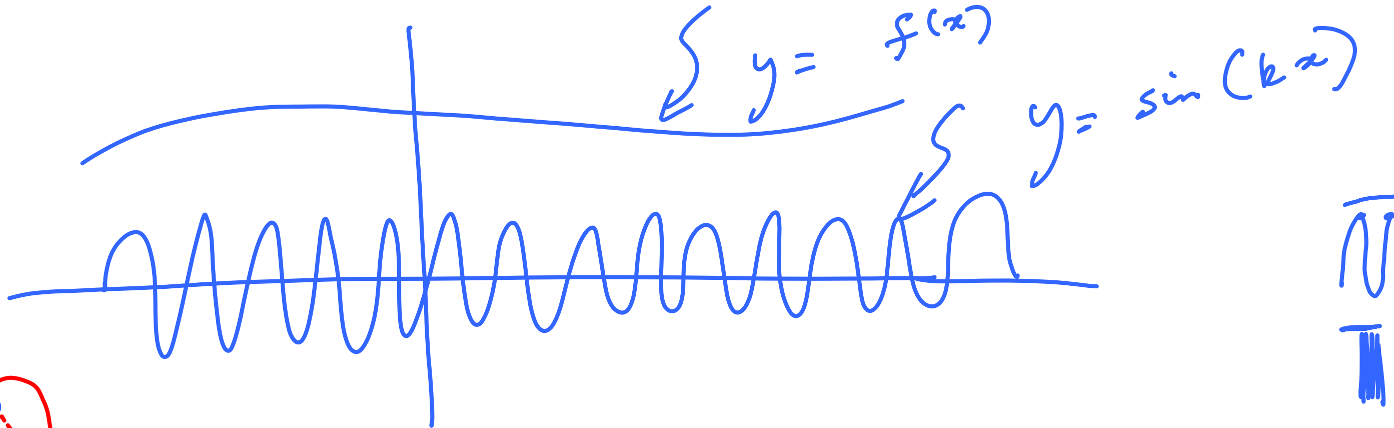
$$s(t) = \begin{cases} t & 0 \leq t \leq \frac{\pi}{2} \\ \pi - t & \frac{\pi}{2} \leq t \leq \pi \end{cases}$$

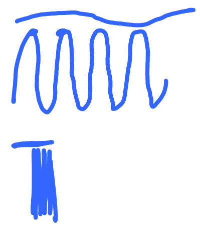
What do you observe ?

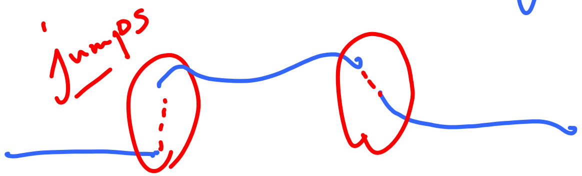
Theorem: Suppose f is a piecewise continuous function on the interval $a \leq x \leq b$, then

$$\lim_{k \to \infty} \int_a^b f(x) \cos(kx)\, dx = \lim_{k \to \infty} \int_a^b f(x) \sin(kx)\, dx = 0$$

Proof:







As the frequency $k \uparrow$, $f(x)$ is nearly constant on two adjacent periods of $\sin(kx)$ and $\cos(kx)$. The integral over each such small period is zero over piecewise intervals.

In the case of a differentiable function f

$$\int_a^b f(x)\cos(kx)\,dx = \frac{\sin(kx)}{k} f(x) \Big|_a^b - \int_a^b \frac{\sin(kx)\, f'(x)}{k}\, dx$$

I II

I : $\dfrac{\sin(kb)f(b) - \sin(ka)f(a)}{k} \to 0$ (Apply l'Hospital's rule)

II $\to 0$ since $f(x) = \text{Const.}$ $\square$

Convergence at a point of Continuity

A Fourier series of f converges to f at a point x if

$$f(x) = a_0 + \lim_{N \to \infty} \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx)$$

Theorem: Suppose f is continuous and a 2π periodic function. Then, at each point 'x' where $f'(x)$ is defined, the F.S. of f at x converges to $f(x)$.

<u>Proof</u> : For a +ve integer N, let

$$S_N(x) = a_0 + \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx)$$

What we need to show :

$$S_N(x) \xrightarrow[N \to \infty]{} f(x)$$

For this, let us rewrite S_N in a slightly different way

STEP 1 : Substituting the Fourier Coeffts,

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\sum_{k=1}^{N}\left(\int_{-\pi}^{\pi} f(t)\cos kt\,dt\right)\cos(kx) + \left(\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt\right)\sin(kx) \quad (1)$$

Using

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$ in (1),

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\frac{1}{2} + \sum_{k=1}^{N}\cos[k(t-x)]\right)dt$$

STEP 2 :

Lemma : For any number $u \in [-\pi, \pi]$

$$\frac{1}{2} + \cos u + \cos(2u) + \dots + \cos(Nu) = \begin{cases} \dfrac{\sin\left(N+\frac{1}{2}\right)u}{2\sin(u/2)} & u \neq 0 \\ \left(N+\frac{1}{2}\right) & u = 0 \end{cases}$$

Proof :

$$\left(e^{ju}\right)^n = \cos(nu) + j\sin(nu)$$

$$\frac{1}{2} + \sum_{k=1}^{N} \cos(ku) = -\frac{1}{2} + \mathrm{Re}\left\{ \sum_{k=0}^{N} \left(e^{ju}\right)^k \right\}$$

$$= -\frac{1}{2} + \mathrm{Re}\left\{ \frac{1 - e^{j(N+1)u}}{1 - e^{ju}} \right\}$$

$$= -\frac{1}{2} + \mathrm{Re}\left\{ \frac{e^{-ju/2} - e^{j(N+\frac{1}{2})u}}{e^{-ju/2} - e^{ju/2}} \right\}$$

$$= -\frac{1}{2} + \mathrm{Re}\left\{ \frac{\cos u/2 - j\sin(u/2) - \left[\cos(N+\frac{1}{2})u + j\sin(N+\frac{1}{2})u\right]}{\cos u/2 - j\sin(u/2) - \left[\cos u/2 + j\sin u/2\right]} \right\}$$

$$= -\frac{1}{2} + \frac{\sin\left(N+\frac{1}{2}\right)u + \sin\left(\frac{u}{2}\right)}{2\sin(u/2)}$$

$$= \begin{cases} \dfrac{\sin\left(N+\frac{1}{2}\right)u}{2\sin(u/2)} & u \neq 0 \\[2ex] N+\frac{1}{2} & u = 0 \end{cases}$$

$\square$

STEP 3 : Let us evaluate the partial sum of the Fourier Series.

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\underbrace{\left[\frac{1}{2} + \sum_{k=1}^{N}\cos[k(t-x)]\right]}_{}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin\left[\left(N+\frac{1}{2}\right)(t-x)\right]}{\sin\left[(t-x)/2\right]}\,dt$$

Define $$P_N(u) = \frac{1}{2\pi}\,\frac{\sin\left(N+\frac{1}{2}\right)u}{\sin\left(u/2\right)}$$

$$S_N(x) = \int_{-\pi}^{\pi} f(t)\, P_N(t-x)\, dt$$

Let $u = t - x$

$$S_N(x) = \int_{-\pi}^{\pi} f(u+x)\, \underbrace{P_N(u)}_{\text{"Kernel"}}\, du$$

(Ponder why the limits on the definite integral remained as is instead of $\pi - x$ & $-\pi - x$)

STEP 4 : Lemma:

$$\int_{-\pi}^{\pi} P_N(u)\, du = 1$$

Proof: Using

$$P_N(u) = \frac{1}{\pi}\left[\frac{1}{2} + \cos u + \cos 2u + \dots + \cos(Nu)\right]$$

$$\Longrightarrow \int_{-\pi}^{\pi} P_N(u)\, du = 1$$

(All the $\cos(ku)$ $k \neq 0$ integrate to zero!)

STEP 5 : Consider $\int_{-\pi}^{\pi} f(u+x)\, P_N(u)\, du$ (From Step 3)

From Step 4 Lemma,

$$f(x) = \int_{-\pi}^{\pi} f(x)\, P_N(u)\, du$$

We are home if we show

$$\int_{-\pi}^{\pi} \left[f(u+x) - f(x) \right] P_N(u)\, du \xrightarrow[N \to \infty]{} 0$$

Now, $\displaystyle \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(u+x) - f(x)}{\sin(u/2)} \sin\left[\left(N+\frac{1}{2}\right)u\right] du$

Invoking prev. theorem, $\displaystyle \lim_{k \to \infty} \int_{-\pi}^{\pi} f(x) \sin(kx) \to 0$

except @ $u = 0, \pi, -\pi$

$$g(u) = \frac{f(u+x) - f(x)}{\sin(u/2)}$$ is continuous except at $u = 0$ over $[-\pi, \pi]$

$$f'(x) = \lim_{u \to 0} \frac{f(u+x) - f(x)}{u}$$ (f' exists by hypothesis)

Consider

$$\lim_{u \to 0} \frac{f(u+x) - f(x)}{u} \cdot \frac{u/2}{\sin(u/2)} \cdot 2$$

$$= f'(x) \cdot 1 \cdot 2 = 2f'(x)$$

Let us define $g(0) = 2f'(x)$, $g(u)$ extends across $u = 0$ as a cont. function

$$\int_{-\pi}^{\pi} [f(u+x) - f(x)] P_N(u)\, du \xrightarrow[N \to \infty]{} 0$$

$\boxtimes$

Convergence at a point of discontinuity

Not all functions are continuous and periodic.
If we consider periodic extensions, it may not necessarily result in a continuous function.

Defn:

$$\lim_{h \to 0+} f(x-h) = f(x-0) \quad \text{(left limit)}$$

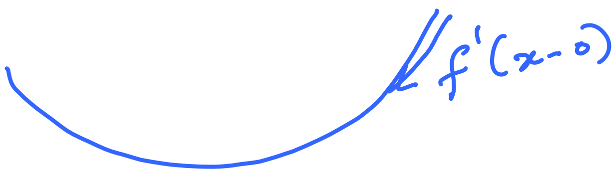
$$\lim_{h \to 0+} f(x+h) = f(x+0) \quad \text{(right limit)}$$

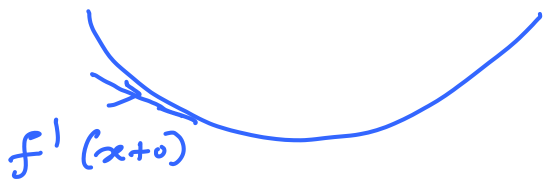
A function f is left differentiable at x if

$$f'(x-0) = \lim_{h \to 0-} \frac{f(x+h) - f(x)}{h}$$

|||ly, a function f is right differentiable @ x if

$$f'(x+0) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$





For $f(x) = x$ over $-\pi \le x < \pi$ & periodic extensions

$f(\pi + 0) = -\pi$ $\quad$ $f(\pi - 0) = \pi$

$f'(\pi - 0) = -1$ $\quad$ $f'(\pi + 0) = 1$

For $f(x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$

$f'(x)$ is not differentiable @ $\pi/2$

Theorem: Suppose $f(x)$ is periodic and piecewise continuous. Suppose x is a point where f is left and right differentiable (but not continuous). The Fourier series of f at x converges to

$$\frac{f(x+0) + f(x-0)}{2}$$

PROOF:

Let us slightly deviate from Step 4 of our previous theorem

$$\int_0^{\pi} P_N(u)\, du = \int_{-\pi}^{0} P_N(u)\, du = \frac{1}{2} \quad \text{—— (1)}$$

Recall: $$P_N(u) = \frac{1}{2\pi} \frac{\sin\left[\left(N + \frac{1}{2}\right) u\right]}{\sin(u/2)}$$ (Even function)

Hence (1)

To prove the theorem, we need to show

$$\int_{-\pi}^{\pi} f(u+x)\, P_N(u)\, du \xrightarrow[N\to\infty]{} \frac{f(x+0)+f(x-0)}{2}$$

$$\int_{0}^{\pi} f(u+x)\, P_N(u)\, du \xrightarrow[N\to\infty]{} \frac{f(x+0)}{2} \qquad \text{———} \quad (A)$$

$$\int_{-\pi}^{0} f(u+x)\, P_N(u)\, du \xrightarrow[N\to\infty]{} \frac{f(x-0)}{2} \qquad \text{———} \quad (B)$$

Using the definition of $P_N(u)$

$$\frac{1}{2\pi}\int_0^{\pi} \frac{f(x+u)-f(x+0)}{\sin(u/2)} \sin\left(N+\frac{1}{2}\right)u \, du \xrightarrow[N\to\infty]{} 0$$

Again $$\frac{1}{2\pi}\int_{-\pi}^{0} \frac{f(x+u)-f(x-0)}{\sin(u/2)} \sin\left(N+\frac{1}{2}\right)u \, du \xrightarrow[N\to\infty]{} 0$$

$\boxtimes$

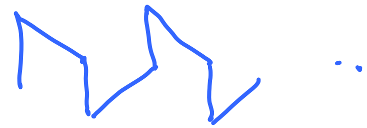
ASIDE : When we have finite such discontinuities such that the measure on that set $\to 0$, we are okay with the convergence proof.

Uniform Convergence

Defn : Given a sequence of functions $\{ f_n(x) \}$, for a given tolerance $\varepsilon > 0$ $\exists$ N /

$$\left| f_n(x) - f(x) \right| < \varepsilon \quad \forall x \text{ and } n \geq N.$$

Defn : A function is piecewise 'smooth' if it is continuous and its derivative is defined everywhere <u>except possibly at a discrete set of points.</u> Ex: Saw tooth waveform



Theorem : The Fourier series of a piecewise smooth 2π periodic function converges uniformly to $f(x)$ on $[-\pi, \pi]$

PROOF : We shall prove this theorem with the assumption that f is everywhere twice differentiable.

Let $$f(x) = \sum_n a_n \cos(nx) + b_n \sin(nx)$$

$$f''(x) = \sum_n a_n'' \cos(nx) + b_n'' \sin(nx) \quad \text{———} \quad (A)$$

Here $a_n'' = -a_n \cdot n^2$; $b_n'' = -b_n \cdot n^2$

Consider $\sum_{n=1}^{\infty} |a_n| + |b_n| = \sum_{n=1}^{\infty} \frac{|a_n''|}{n^2} + \frac{|b_n''|}{n^2}$ —— B

If f'' is continuous, then a'' & b'' stay bounded by a quantity say 'M' and 'N'

$\therefore$ B can be written as an inequality

$$\leq \sum_{n=1}^{\infty} \frac{M}{n^2} + \frac{N}{n^2}$$

$$= (M+N) \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{finite \& bounded } \frac{\pi^2}{6}}$$

To ensure uniform conv., we need another results.

Lemma: Suppose

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

with $\sum_{k=1}^{\infty} |a_k| + |b_k| < \infty$, then

F.S. converges uniformly & absolutely to the function.

Proof: By triangular inequality,

$$|a_k \cos(kx) + b_k \sin(kx)| \leq |a_k| + |b_k|$$

$$\left(\because \quad |\cos(\cdot)| \ \& \ |\sin(\cdot)| \leq 1\right)$$

Let $S_N(x) = a_0 + \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx)$

$$f(x) - S_N(x) = \sum_{k=N+1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

$$\left| f(x) - S_N(x) \right| \leq \sum_{k=N+1}^{\infty} |a_k| + |b_k|$$

But, with our <u>twice</u> differentiability condition,

$$\sum_{k=N+1}^{\infty} |a_k| + |b_k| < \infty$$

$\therefore$ For a given $\varepsilon > 0$ $\exists$ $N_0 > 0$ so that

$$N > N_0 \implies \sum_{k=N+1}^{\infty} |a_k| + |b_k| < \varepsilon$$ irrespective of 'x'

$\implies$ Uniform Convergence $\blacksquare$

Use Lemma, to conclude the Theorem. $\blacksquare$